Higher Coleman theory
w/ Vincent Pilloni

Case of modular curve

- $X_I = X_0(p)/\mathbb{Z}_p$ modular curve
  moduli of (E, C)
  $C \subseteq E[p]$ order p

  I = Iwahori
  $= \left\{ \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \right\} \subseteq GL_2(\mathbb{Q}_p)$

- ω / X_I modular line bundle

  $\omega = e^* \Omega^1_{\mathcal{E}/X_I}$ $\quad \begin{array}{c} \mathcal{E} \\ e \curvearrowleft \downarrow \\ X_I \end{array}$ univ EC

- $\mathcal{X}_I = (X_{I,\mathbb{Q}_p})^{ad}$ adic modular curve

$k \in \mathbb{Z}$

$H^0(X_I, \omega^k)$ Classical wt k modular forms ($= 0$ $k < 0$)

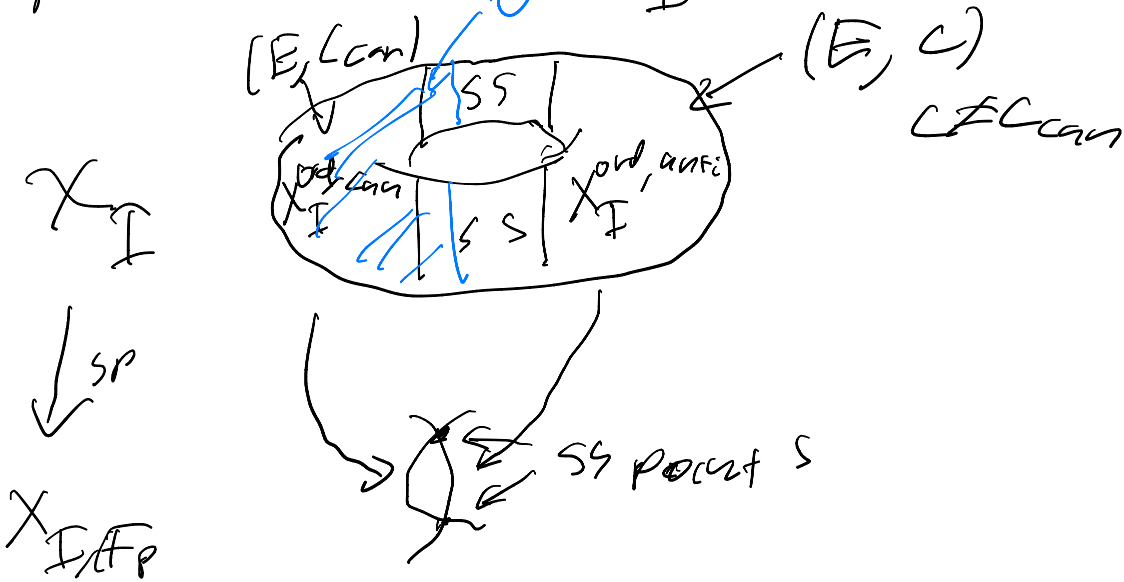
We can also look at

$$H^1(X_I, \omega^k) \simeq H^0(X_I, \omega^{2-k}(-\mathrm{cusps}))^\vee$$

Serre duality + Kodaira-Spencer

<u>Basic problem</u>: develop a theory of p-adic modular forms for H^1

Picture of X_I:



E EC good ordinary reduction

$$C_{can} = \ker(E[p] \to \underbrace{E[p]}_{\text{ord. } p})$$

We can take a strict nbhd

$$\overline{X_I^{ord,can}} \subseteq V$$

can consider $H^0(V, \omega^k)$

Basic observation: the restriction map sits in a 4 term exact seq

$$0 \to H^0(X_I, \omega^k) \to H^0(V, \omega^k) \to H^1_Z(X_I, \omega^k) \to H^1(X_I, \omega^k) \to 0$$

$$Z = X_I - V$$

(Reminder: $R\Gamma_Z$ is derived functor of Γ_Z of sections supported in Z

in general there is a triangle

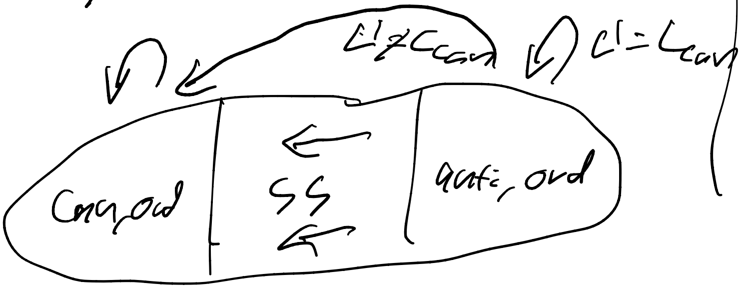
$$R\Gamma_Z(X, -) \to R\Gamma(X, -) \to R\Gamma(U, -) \xrightarrow{+1}$$

Need to bring in

$$U_p = [I \begin{pmatrix} 1 & \\ & p \end{pmatrix} I]$$

$$U_p(E, C) = \{(E/C', E[p]/C') \mid C' \neq C\}$$

Dynamics of U_p



If E ord, $C' \neq C_{can}$, $E[p]/C'$ is canonical

We need a way to measure when we are on ss locus

- Could use $v(\mathrm{Ha})$ or Fargues deg function
- Could use π_{HT}

$$\begin{tikzcd}
(E, \mathbb{Z}_p^2 \simeq T_pE) \quad \mathcal{X}_\infty \arrow[r, "\pi_{HT}"] \arrow[d] & \mathbb{P}^1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \backslash GL_2 \arrow[d] \\
(E, C) \quad \mathcal{X}_I = \mathcal{X}_\infty / I \arrow[r, "\pi_{HT, I}"] & \mathbb{P}^1 / I
\end{tikzcd}$$

$\uparrow$ $GL_2(\mathbb{Q}_p)$-eq

$\uparrow$ $[I \begin{pmatrix} 1 & \\ & p \end{pmatrix} I]$-equivariant

$\leftarrow$ just a top space

$$(E, \mathbb{Z}_p^2 \simeq T_p E) / \mathrm{Spa}(R, R^+)$$

$$\mathbb{Z}_p^2 \otimes R \simeq T_p E \otimes_{\mathbb{Z}_p} R \twoheadrightarrow \omega_E$$

R point of $\mathbb{P}^1$

$$\overline{\mathcal{X}_I^{\mathrm{can,ord}}} = \pi_{HT,I}^{-1}([\overset{0}{0,1}] \cdot I)$$

$$\overline{\mathcal{X}_I^{\mathrm{anti,ord}}} = \pi_{HT,I}^{-1}([\overset{\infty}{1,0}] \cdot I)$$

For $\alpha \in \mathbb{Q}$ let $U_\alpha \subseteq \mathbb{P}^1$ be the open Tate disc around 0 of radius $p^{-\alpha}$

Let $Z_\alpha = \mathbb{P}^1 - U_\alpha$

Dynamical facts:

$$U_p(U_\alpha \cdot I) \subseteq U_{\alpha+1} \cdot I \qquad \alpha \geq 0$$

$$U_p^t(Z_\alpha \cdot I) \subseteq Z_{\alpha-1} I \qquad \alpha \leq 0$$

$$\uparrow$$

$$\left[I \begin{pmatrix} 1 & \\ & p \end{pmatrix} I\right]$$

Can define some cohomologies

$$H^0_{1,\alpha}(k) = H^0(\pi^{-1}_{HT,I}(U_\alpha \cdot I), \omega^k) \qquad \alpha \geq 0$$

$$H^1_{w,\alpha}(k) = H^1_{\pi^{-1}_{HT,I}(Z_\alpha \cdot I)}(X_I, \omega^k) \qquad \alpha \leq 0$$

Have 4-term sequence

$$0 \to H^0(X_I, \omega^k) \to H^0_{1,0}(k) \to H^1_{w,0}(k) \to H^1(X_I, \omega^k)$$

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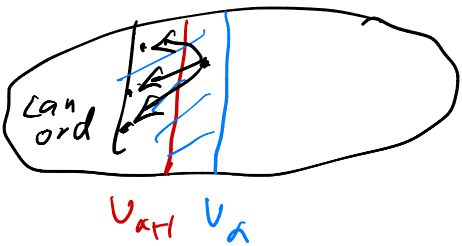
Analytic Continuation:

$$\begin{tikzcd} H^0_{1,\alpha+1}(k) \arrow[r, "U_p"] \arrow[rr, bend right, "U_p"'] & H^0_{1,\alpha}(k) \arrow[loop above, "U_p"] \arrow[r, "\mathrm{res}"] & H^0_{1,\alpha+1}(k) \end{tikzcd}$$

Compact



U_p is compact on $H^0_{1,\alpha}(k)$

$\rightsquigarrow$ can form $H^0_{1,\alpha}(k)^{fs}$ $\leftarrow$ finite slope

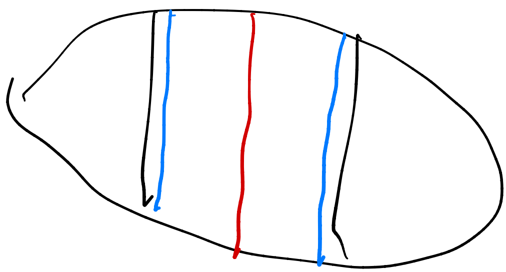
i.e. alg direct sum of gen eigenspaces for U_p w/ nonzero eigenvalue

and $H^0_{1,\alpha}(k)^{fs}$ are all isomorphic

call it $H^0_1(k)^{fs}$

Similarly $H^1_{w,\alpha}(k)^{fs}$ are all isomorphic, call it $H^1_w(k)^{fs}$

$$0 \to H^0(X_{I_f}, \omega^k)^{(fs)} \to H^0_1(k)^{fs} \to H^1_w(k)^{fs} \to H^1(X_{I_f}, \omega^k)^{fs} \to$$



Classicality:

Prop:

Valuations of eigenvalues

- The slopes of U_p on $H^0_1(k)^{fs}$ are ≥ 1
- The slopes of U_p on $H^1_w(k)^{fs}$ are $\geq k$

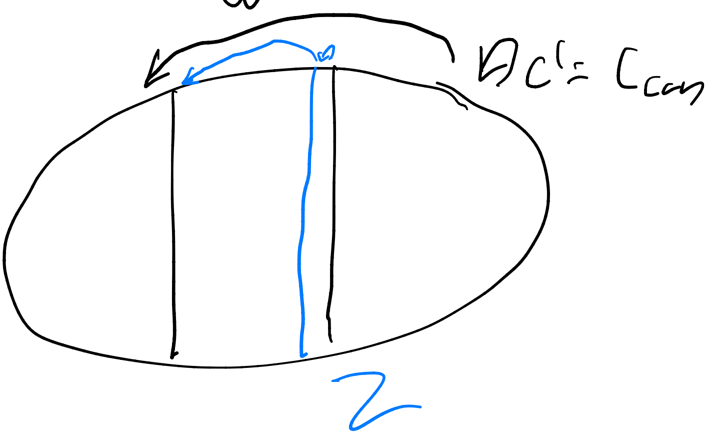
Cor (Coleman Classicality)

- $H^0(X_I, \omega^k)^{<k} \xrightarrow{\sim} H^0_1(k)^{<k}$
- $H^1_w(k)^{<1} \xrightarrow{\sim} H^1(X_I, \omega^k)^{<1}$

Idea of pf of prop:

For $H^0_1(k)$ restrict to ordinary locus and use q-expansions or Serre-Tate coords

For $H^1_w(k)$



$E \longrightarrow E/C_{can}$

ordinary $\uparrow$

Probably mod p

$W_{E/C_{can}} \longrightarrow W_E$

$\cup$

$W_{E/C_{can}} \longrightarrow p W_E$ $1-E$

on 2

General Shimura varieties

- (G, X) Shimura datum of AL type
- p prime $G_{\mathbb{Q}_p}$ quasisplit
- $T \subseteq B \subseteq P_N \subseteq G_{\mathbb{Q}_p}$ $\quad P_N \to M_N$ Levi

  $\uparrow$ parabolic associated to N
- $Fl = P_N \backslash G_{\mathbb{Q}_p}$
- W absolute Weyl group of G

  $\cup$

  ${}^M W$ set of minimal length reps $W_M \backslash W$

- $I \subseteq G(\mathbb{Q}_p)$ Iwahori
- $T^+ = \{ t \in T(\mathbb{Q}_p) \mid v(\alpha(t)) \geq 0 \quad \forall \alpha \in \Phi^+ \}$

  $\cup$

  $T^{++} = \quad " \qquad\qquad > 0$

- $H_I = C_c(I \backslash G(\mathbb{Q}_p) / I, \mathbb{Q})$

  $\cup$

  $H^+ = \text{span}\,[ItI]$ for $t \in T^+$

  $\cong$

  $\mathbb{Q}[T^+]$

From now on "finite slope" will always mean for H^+

$\rightsquigarrow$ Shimura variety S_I

By C-S we have a diagram

$$\begin{tikzcd} \text{perfectoid Shimura var} \quad S_\infty \arrow[r, "\pi_{HT}"] \arrow[d] & \mathcal{F}\ell \arrow[d] \\ S_I = S_\infty / I \arrow[r, "\pi_{HT, I}"] & \mathcal{F}\ell / I \end{tikzcd}$$

Our interpretation of M_W is that it is $\mathrm{Fix}_T(\mathcal{F}\ell)$

$$\begin{matrix} M_W & \ni & w \\ \mathrm{Fix}_T(\mathcal{F}\ell) & \ni & P_\mu w \end{matrix}$$

We will consider the regions

$\pi_{HT, I}^{-1}(P_\mu \cdot w \cdot I)$ for $w \in M_W$

For $G = GL_2$ there are exactly canonical / anticanonical ord loci

More generally if $G_{\mathbb{Q}_p}$ is split, they are again components of ordinary locus $w \Leftrightarrow$ rel pos of level struc and canonical subsp

But in general they can be non ordinary (non μ-ordinary)

<u>Ex</u>: $G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$ F real quad field

p inert in F

$W = {}^1W = \{1, w_0\}^2$

$(1,1), (w,w) \longleftrightarrow$ can / ansic ord locus

$(1,w), (w,1) \longleftrightarrow$ Locus where $A[p^\infty] \simeq LT_1 \times LT_2$

Coherent Cohomology

$\kappa \in X^*(T)^{M, +} \rightsquigarrow \mathcal{V}_\kappa / S_I$ automorphic vector bundle

we want to study $R\Gamma(S_I, \mathcal{V}_\kappa)$

These can be computed in terms of automorphic forms (Harris, Su, ...)

π aut rep for $G(\mathbb{A})$ contributes according to π_∞

The tempered Π_∞ which contribute are "nondegenerate limits of discrete series"

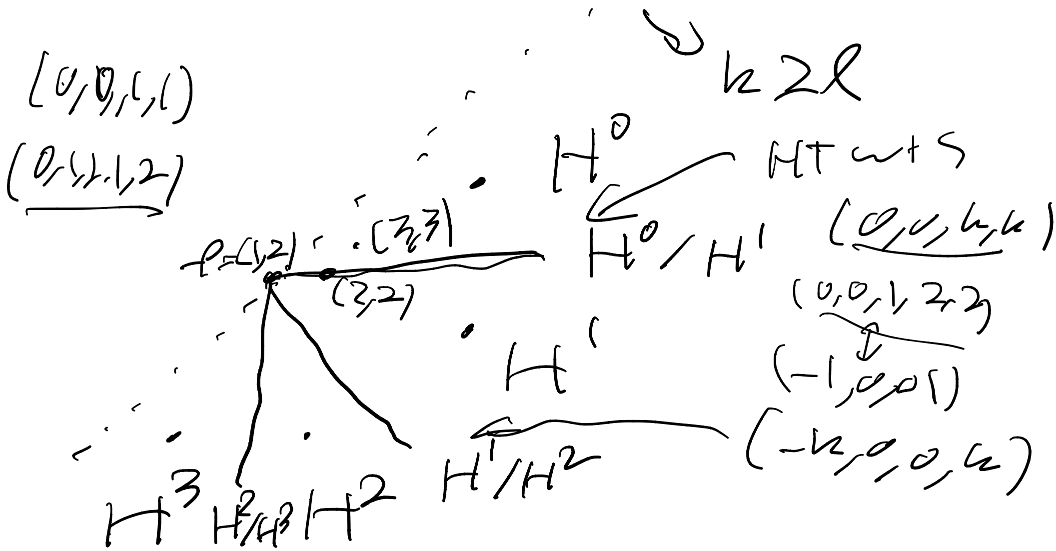
An L-packet is associated to a $\nu \in X^*(T)$ w/ $\underline{\nu+\rho} \in X^*(T)^+_{\mathbb{R}}$

infinitesimal character

The L-packet contributes to

$H^{\ell(w)}(S_{\mathcal{I}}, V_{\kappa_w})$ for $w \in {}^M W$ such that $\kappa_w = -w_{0,M} w(\nu+\rho) + \rho$ is M-dominant

The picture for $G = GSp_4$



$K = (k, \ell) \rightsquigarrow V_K = \mathrm{Sym}^{k-\ell}\Sigma \otimes \det^{\ell}\Sigma$

An overview of Higher Coleman Theory.

- Construct for each $w \in {}^M W$ an "overconvergent cohomology" $R\Gamma_w(K)^{fs}$

by taking cohomology w/ support of V_K in some nbhd of $\pi_{HT,I}^{-1}(P_\mu \cdot w \cdot I)$

- Construct a spectral sequence

$$E_1^{pq} = \bigoplus_{\substack{w \in {}^M W \\ \ell(w) = p}} H_w^{p+q}(K) \Rightarrow H^{p+q}(S_I, V_K)$$

when $G = GL_2$ it is just the 4 term exact sequence.

- We prove a lower bound on slopes. For $w \in {}^M W$, $t \in T^+$ the slopes $[I+I]$ on $R\Gamma_w(K)$ are $\geq v((w^{-1} w_{0,M} K)(t))$

- (1st classicality thm ("geometric")) If $K + \rho$ regular and $w \in {}^M W$ is s.t. $-w^{-1} w_{0,M} K \in X^*(T)^+$

Then

$$R\Gamma(S_{I}, \mathcal{V}_{K})^{ss^{M}(K)} \simeq R\Gamma_{w}(K)^{ss^{M}(K)}$$

<u>Step 2</u>: p-adically vary the sheaf

For each $w \in {}^{M}W$, $\nu : T(\mathbb{Z}_p) \to \mathbb{C}_p^{\times}$ we construct "locally analytic overconvergent cohomology"

$R\Gamma_{w,an}(\nu)^{fs}$

This is cohomology of a big sheaf $\mathcal{V}_{\nu}$ w/ same support conditions as in def of $R\Gamma_{w}$

· For $\kappa \in X^*(T)^{M,+}$

Let $\nu = -w'w_{0,M}(\kappa+\rho)-\rho$

we get a map of sheaves

$$\mathcal{V}_\kappa \longrightarrow \mathcal{V}_\nu$$

which looks locally like the inclusion of alg induction into loc analytic induction.

$$\rightsquigarrow R\Gamma_w(\kappa)^{fs} \longrightarrow R\Gamma_{w,an}(\nu)^{fs}$$

2nd classicality thm

$$R\Gamma_w(\kappa)^{ss_M(\kappa)} \xrightarrow{\sim} R\Gamma_{w,an}(\nu)^{ss_M(\kappa)}$$

Now $R\Gamma_{v,an}(\mathcal{V})$ vary p-adically

$\rightsquigarrow$ construct Eigenvalues