Certain representations with unique models

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- $\mathbb{H}$: the quaternion algebra over $\mathbb{R}$
- $\nu : \mathbb{H}^\times \to \mathbb{R}_{>0}$
- $\mathrm{tr} : \mathbb{H} \to \mathbb{R}$
- $\psi : \mathbb{R} \to \mathbb{C}^\times$ nontrivial additive character
- $N = \left\{ u = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{H}) \right\}.$
- $\psi_N(u) = \psi(\mathrm{tr}(x))$

- π: an irreducible 5-dimensional representation of $\mathbb{H}^\times$
- the normalized parabolic induction

$$\pi \times \nu\pi$$

  has a unique irreducible subrepresentation $\theta(\pi)$.

Question

dim $\mathrm{Hom}_N(\theta(\pi), \psi_N) =?$

A 25

B 10

C 1

D 0

Uniqueness of Whittaker models, I

- F: non-Archimedean local field
- $\psi : F \to \mathbb{C}^\times$, a nontrivial additive character
- GL_n (more generally, quasi-split groups)
- Write GL_n for $\mathrm{GL}_n(F)$
- $\nu = |\det| : \mathrm{GL}_n \to \mathbb{C}^\times$
- $$N_n = \left\{ u = \begin{pmatrix} 1 & u_{12} & * & \cdots & * \\ & 1 & u_{23} & \cdots & * \\ & & 1 & \cdots & * \\ & & & & \vdots \\ & & & & 1 \end{pmatrix} \in \mathrm{GL}_n \right\}.$$

  A generic character $\psi_n : N_n \to \mathbb{C}^\times$ is of the form

  $$\psi_n(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n}).$$

Uniqueness of Whittaker models, II

Theorem (Uniqueness of Whittaker models)

For $\pi \in \mathrm{Irr}(\mathrm{GL}_n)$,

$$\mathrm{Hom}_{N_n}(\pi, \psi_n) = \mathrm{Hom}_{\mathrm{GL}_n}(\pi, \mathrm{ind}_{N_n}^{\mathrm{GL}_n} \psi_n)$$

is of dimension ≤ 1. Equivalently,

$$\dim J_{N_n, \psi_n}(\pi) \leq 1.$$

When the dimension is 1, we say that π is generic (or ψ_N-generic) or π has a Whittaker model.

Uniqueness of Whittaker models, III

Applications

- Such properties play important roles in the construction of many global integrals. (Use unique models to obtain Eulerian integrals.)
- Can be used to study the analytic properties of certain Langlands L-functions.
- For example, the Rankin-Selberg integrals and Langlands-Shahidi method.

Non-generic representations

When π does not have any Whittaker model, we say that π is non-generic.

Degenerate models

Non-generic representations admit unique models of degenerate type.

Derivatives, I

- Mirabolic subgroup
$$P_n = \left\{ \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} : g \in \mathrm{GL}_{n-1}, v \in F^{n-1} \right\}.$$
- $$U_n = \left\{ \begin{pmatrix} I_{n-1} & v \\ 0 & 1 \end{pmatrix} : v \in F^{n-1} \right\}.$$
- $P_n = \mathrm{GL}_{n-1} \ltimes U_n$
- the restriction of ψ_n gives a character of U_n

Derivatives, II

Several functors

- $\Psi^{-}(\pi) = J_{U_n}(\pi) = \pi/\langle \pi(u)v - v : u \in U_n, v \in \pi \rangle$. This gives
$$\Psi^{-} : \mathrm{Rep}(P_n) \to \mathrm{Rep}(\mathrm{GL}_{n-1}).$$
- $\Phi^{-}(\pi) = J_{U_n,\psi_n}(\pi) = \pi/\langle \pi(u)v - \psi_n(u)v : u \in U_n, v \in \pi \rangle$ and this gives
$$\Phi^{-} : \mathrm{Rep}(P_n) \to \mathrm{Rep}(P_{n-1}).$$
- k-th derivative
$$\pi^{(k)} = \Psi^{-} \circ (\Phi^{-})^{(k-1)}(\pi|_{P_n}).$$
This gives a functor
$$\mathrm{Rep}(\mathrm{GL}_n) \to \mathrm{Rep}(\mathrm{GL}_{n-k}).$$

Derivatives, III

- The n-th derivative is the functor J_{N_n,ψ_n}.
- Let k_0 be the maximal k such that $\pi^{(k)} \neq 0$. Then $\pi^{(k_0)}$ is called the highest derivative of π. Notation: $k_0 = ht(\pi)$.
- If π is generic, then the highest derivative of π is the n-th derivative.

Derivatives, IV

Example (Speh representations)

If $\tau \in \mathrm{Irr}(\mathrm{GL}_n)$ is discrete series, then the normalized parabolic induction

$$\tau \times \tau\nu \times \cdots \times \tau\nu^{\ell-1}$$

has a unique irreducible subrepresentation $\theta(\tau, \ell) \in \mathrm{Irr}(\mathrm{GL}_{n\ell})$.

In particular, if $\tau : \mathrm{GL}_1 \to \mathbb{C}^\times$ is a character, then

$$\theta(\tau, \ell) = \tau \circ \det .$$

Generally

If $\tau \in \mathrm{Irr}(\mathrm{GL}_n)$ is generic and unitary, then $\tau = \tau_1 \times \cdots \times \tau_m$ for $\tau_1, \cdots, \tau_m$ essentially discrete series. Define

$$\theta(\tau, \ell) = \theta(\tau_1, \ell) \times \cdots \times \theta(\tau_m, \ell).$$

Derivatives, V

- the highest derivative of $\theta(\tau, \ell)$ is $\theta(\tau, \ell)^{(n)}$ "$=$" $\theta(\tau, \ell - 1)$.

More generally,

Theorem (Zelevinsky)

If π is irreducible, then its highest derivative $\pi^{(k)}$ is also irreducible.

Derivatives, VI

Given $\pi \in \mathrm{Irr}(\mathrm{GL}_n)$, we can take highest derivatives repeatedly:

$$\begin{aligned} k_1 &= ht(\pi), & \pi_1 &= \pi^{(k_1)}, \\ k_2 &= ht(\pi_1), & \pi_2 &= \pi_1^{(k_2)}, \\ &\cdots \\ k_m &= ht(\pi_{m-1}), & \pi_m &= \pi_{m-1}^{(k_m)}. \end{aligned}$$

This gives a partition $(k_1 k_2 \cdots k_m)$ of n.

- π_m is of the form $J_{N_n,\psi_{(k_1\cdots k_m)}}(\pi)$ for some degenerate character $\psi_{(k_1\cdots k_m)}$.
- By the Frobenius reciprocity, this gives a degenerate model for π.
- By the theorem of Zelevinsky, π_m is an irreducible representation of GL_0, which must be one-dimensional.

Nilpotent orbits, I

Summary:

- by computing derivatives, one can find a partition $(k_1 k_2 \cdots k_m)$ and a unique model for π.
- $(k_1 k_2 \cdots k_m)$ is the "maximal" partition (or nilpotent orbit) that support nonzero models for π.

Nilpotent orbits, II

More generally, given a reductive group G, to every coadjoint nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ and every $\pi \in \mathrm{Rep}(G)$, we associate a certain generalized Whittaker quotient $\pi_{\mathcal{O}}$.

- Let $\mathrm{WO}(\pi)$ denote the set of all nipotent orbit $\mathcal{O}$ with $\pi_{\mathcal{O}} \neq 0$
- $\mathrm{WS}(\pi)$ denote the set of maximal orbits in $\mathrm{WO}(\pi)$ with respect to the closure ordering.

Example

- Nilpotent orbits of GL_n are classified by the partitions of n via the Jordan canonical decomposition.
- $\mathrm{WS}(\theta(\tau, \ell)) = \{(n^\ell)\}$.

Nilpotent orbits, III

Character expansion

One can define the character χ_π of π as a distribution and we have a charater expansion

$$\chi_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}.$$

where the sum is over the set of nilpotent orbits.

Theorem (Mœglin-Waldspurger, Varma)

The set $\mathrm{WS}(\pi)$ is the same as the maximal elements such that $c_{\mathcal{O}} \neq 0$.
Moreover, for $\mathcal{O} \in \mathrm{WS}(\pi)$, $\dim \pi_{\mathcal{O}} = c_{\mathcal{O}}$.

Nilpotent orbits, IV

Example

For $\theta(\tau, \ell)$, $c_{(n^\ell)} = 1$ and

$$\chi_{\theta(\tau,\ell)} = \hat{\mu}_{(n^\ell)} + \text{ other terms.}$$

Archimedean case

- There are irreducible representations without unique models.
- The Archimedean version of Mœglin-Waldspurger's theorem has not been proven.

Division algebras, I

- D: central division algebra over F of dimension d^2
- Consider $\mathrm{GL}_{n,D}$
- Nilpotent orbits of $\mathrm{GL}_{n,D}$ are classified by partitions of n. Notation: $(n_1 \cdots n_m)_D$.

Unique models?

Unfortunately, uniqueness of models fails in general.

Division algebras, II

Question

Find representations of $\mathrm{GL}_{n,D}$ with unique models.

Example (Case $n = 1$)

There is no non-trivial nilpotent elements in $D^\times$ but there are irreducible finite-dimensional representations of $D^\times$ of dimension greater than 1.
Only one-dimensional representations have unique models.

Jacquet-Langlands correspondence, I

How to construct representations of $\mathrm{GL}_{n,D}$?

- For $g' \in \mathrm{GL}_{n,D}$, one can define characteristic polynomial
- $g \in \mathrm{GL}_{nd}$, $g' \in \mathrm{GL}_{n,D}$
- Define: $g \leftrightarrow g'$ if and only if g and g' are both regular semi-simple and have the same characteristic polynomials.
- $\mathcal{O} = (n_1^d \cdots n_m^d)$ in $\mathfrak{gl}_{nd}^*$ corresponds to $\mathcal{O}' = (n_1 \cdots n_m)_D$ in $\mathfrak{gl}_{n,D}^*$
- $\mathcal{D}_n$: discrete series of GL_n
- $\mathcal{D}'_n$: discrete series of $\mathrm{GL}_{n,D}$

Jacquet-Langlands correspondence, II

Theorem (Deligne-Kazhdan-Vignéras)

There is a unique bijection $\mathsf{C} : \mathcal{D}_{nd} \to \mathcal{D}'_n$ such that for all $\pi \in \mathcal{D}_{nd}$ we have

$$\chi_\pi(g) = (-1)^{nd-n} \chi_{\mathsf{C}(\pi)}(g')$$

for all $g \in \mathrm{GL}_{nd}$ and $g' \in \mathrm{GL}_{n,D}$ such that $g \leftrightarrow g'$.

Theorem (Badulescu, Badulescu-Renard)

If π is a '*d*-compatible' irreducible unitary representation of GL_{nd}, then there exists a unique irreducible unitary representation π' of $\mathrm{GL}_{n,D}$ and a unique sign $\varepsilon_\pi \in \{-1, 1\}$ such that

$$\chi_\pi(g) = \varepsilon_\pi \chi_{\pi'}(g')$$

for all $g' \leftrightarrow g$. Notation: $\pi' = \mathsf{LJ}(\pi)$.

Jacquet-Langlands correspondence, III

We will take the later version as it is compatible with a global correspondence.

Non-Archimedean Strategy

- (Prasad's result) character relation implies identities $c_{\mathcal{O}} = \varepsilon_\pi c_{\mathcal{O}'}$, where $\mathcal{O} \subset \mathfrak{gl}_{nd}^*$ corresponds to $\mathcal{O}' \subset \mathfrak{gl}_{n,D}^*$.
- Idea: find representations of GL_{nd} with suitable size such that $\mathcal{O} \in \mathrm{WS}(\pi)$ corresponds to $\mathcal{O}' \in \mathrm{WS}(\mathrm{LJ}(\pi))$.
- (Important!) find representations such that $\varepsilon_\pi = 1$

Jacquet-Langlands correspondence, IV

Definition

For a positive integer ℓ and an irreducible generic unitary τ, define

$$\theta_D(\tau, \ell) = \mathsf{LJ}(\theta(\tau, d\ell)).$$

- $\theta(\tau, d\ell)$ is d-compatible.
- $\varepsilon_{\theta(\tau, d\ell)} = 1$
- $\mathrm{WS}(\theta(\tau, d\ell)) = (n^{d\ell})$
- one can check that $\mathrm{WS}(\theta_D(\tau, \ell)) = (n^{\ell})_D$ with unique models.
- If τ is one-dimensional, then $\theta(\tau, d\ell) = \tau \circ \det$ and $\theta_D(\tau, \ell) = \tau \circ \mathrm{Nm}$.

Jacquet-Langlands correspondence, V

- D: unique quaternion algebra over F
- π: Steinberg representation of GL_2.
- 1_{GL_2}, $1_{D^\times}$: trivial representations

Then

- $\mathsf{C}(\pi) = 1_{D^\times}$, but

$$\chi_\pi(g) = -\chi_{1_{D^\times}}(g') \text{ for all } g \leftrightarrow g'.$$

- $\mathsf{LJ}(1_{\mathrm{GL}_2}) = 1_{D^\times}$ and

$$\chi_{1_{\mathrm{GL}_2}}(g) = \chi_{1_{D^\times}}(g') \text{ for all } g \leftrightarrow g'.$$

Jacquet-Langlands correspondence, VI

Archimedean case

The definition of $\theta_{\mathbb{H}}(\tau, \ell)$ works. Similar results are expected but a different approach is required.

Global definition

Given a cuspidal representation $\tau = \otimes'_v \tau_v$ of $\mathrm{GL}_n(\mathbb{A})$, one can define

$$\theta_D(\tau, \ell) = \otimes'_v \theta_{D_v}(\tau_v, \ell),$$

and this is a discrete series of $\mathrm{GL}_{n\ell, D}(\mathbb{A})$.

One can ask similar questions for global representations (in terms of degenerate Whittaker coefficients).

Note: for central simple algebra $D_v = \mathrm{M}_{r_v}(A_v)$,

$$\theta_{D_v}(\tau_v, \ell) = \theta_{A_v}(\tau_v, r_v \ell).$$

Archimedean case, I

Can be reduced to the case τ discrete series.
Let $\tau \in \mathcal{D}(\mathrm{GL}_2(\mathbb{R}))$ and let $\tau' = \mathsf{C}^{-1}(\tau) \in \mathrm{Irr}(\mathbb{H}^\times)$. Assume that $\dim \tau' > 1$.

The representation

Then $\theta_{\mathbb{H}}(\tau, \ell)$ is the unique irreducible subrepresentation of the parabolic induction

$$\tau'\nu^{(1-\ell)/2} \times \tau'\nu^{(3-\ell)/2} \times \cdots \times \tau'\nu^{(\ell-1)/2}$$

where $\nu : \mathbb{H}^\times \to \mathbb{R}_{>0}$ is the reduced norm.

Then, $\mathrm{WS}(\theta_{\mathbb{H}}(\tau, \ell)) = (2^\ell)_{\mathbb{H}}$ with unique model.

Archimedean case, II

The first known result was the case $\ell = 1$.

The case $\ell = 1$

The representation $\theta_{\mathbb{H}}(\tau, 1)$ is the unique irreducible subrepresentation of $\tau'\nu^{-1/2} \times \tau'\nu^{1/2}$ and

$$\dim \mathrm{Hom}_{N_{(2)_{\mathbb{H}}}}(\theta_{\mathbb{H}}(\tau, 1), \psi_{(2)_{\mathbb{H}}}) = 1$$

where

$$N_{(2)_{\mathbb{H}}} = \left\{ u = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\} \text{ and } \psi_{(2)_{\mathbb{H}}}(u) = \psi(\mathrm{tr}(x)).$$

Archimedean case, III

Hang Xue's idea

The construction

$$\tau \mapsto \theta_{\mathbb{H}}(\tau, 1)$$

can be realized as the theta correspondence from $\mathrm{SL}_2 \to \mathrm{SO}(5,1)$.

Gomez-Zhu's result

There is an isomorphism between the Whittaker model for SL_2 and the $(2)_{\mathbb{H}}$-model of $\theta_{\mathbb{H}}(\tau, 1)$.

How about the case of general ℓ? (This can be proved using a global method.)

Kirillov models, I

Statement

For a generic representation π of GL_n

$$\mathrm{ind}_{N_n}^{P_n} J_{N_n,\psi_n}(\pi) \ltimes \psi_n \hookrightarrow \pi|_{P_n}.$$

Representation theory of P_n

The group P_n is the semi-direct $\mathrm{GL}_{n-1} \ltimes U_n$. The irreducible representations of P_n is classified by

- A orbit $\mathrm{GL}_{n-1} \cdot X$ of $\widehat{U}_n$ under the action of GL_{n-1} (only two orbits)
- An irreducible representation τ_X of the stabilizer M_X of ψ_X in GL_{n-1}.

The construction is given by $\mathrm{ind}_{M_X \ltimes U_n}^{P_n}(\tau_X \ltimes \psi_X)$.

Kirillov models, II

Observe that

- representations coming from different orbits are not isomorphic.
- As a result, the Kirillov model captures the generic part of of $\pi|_{P_n}$
- the Kirillov model is a supercuspidal representation.

For a simple division algebra D, one can introduce $P_{n,D}, N_{n,D}, \psi_{n,D}, U_{n,D}$ etc. The theory of Kirillov models extends to representations of $\mathrm{GL}_{n,D}$.

The global case

The general case can be reduced to case $\ell = 1$ by induction in stages.

the case $\ell = 1$

Show that, for some $\varphi \in \theta_D(\tau, 1)$,

$$W_\varphi(g) := \int_{N_{n,D}(F)\backslash N_{n,D}(\mathbb{A})} \varphi(ug)\psi_{n,D}(u)\, du \neq 0.$$

In other words, $\theta_D(\tau, 1)$ is "D-generic".

(We use ideas of Kazhdan-Patterson 1984.)

Note

If $D = F$, the argument below shows the following:
Let τ be an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. If τ_{v_0} is a generic representation for a non-Archimedean place v_0, then τ is globally generic.

- Fix a non-Archimedean place v_0, we already know that $\theta_D(\tau,1)_{v_0}$ is "D_{v_0}-generic", and therefore has a Kirillov model $\mathcal{K}_{v_0} \hookrightarrow \theta_D(\tau,1)_{v_0}$. It is "$D_{v_0}$-cuspidal".
- Consider the $P_{n,D}(\mathbb{A})$-representation

$$T := \mathcal{K}_{v_0} \otimes (\otimes'_{v\neq v_0}\theta_D(\tau,1)_v) \subset \otimes'_v \theta_D(\tau,1)_v.$$

  This is a cuspidal representation.
- Fourier expansion. For $\varphi \in T$ and $g \in P_{n,D}(\mathbb{A})$

$$\varphi(g) = \sum_{\gamma \in N_{n-1,D}(F)\backslash \mathrm{GL}_{n-1,D}(F)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

- $\varphi|_{P_{n,D}(F)\backslash P_{n,D}(\mathbb{A})} \neq 0$ since $Z_{n,D}P_{n,D}(F)\backslash Z_{n,D}P_{n,D}(\mathbb{A})$ is dense in $\mathrm{GL}_{n,D}(F)\backslash \mathrm{GL}_{n,D}(\mathbb{A})$.
- One of $W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \neq 0$.

Archimedean case, IV

We are now back to the Archimedean case.

- $\tau_\infty \in \mathcal{D}(\mathrm{GL}_2(\mathbb{R}))$
- Embed τ_∞ as the Archimedean component of $\tau \in \mathrm{Cusp}(\mathrm{GL}_2(\mathbb{A}))$. (May assume $F = \mathbb{Q}$).
- Then $\theta_{D_\infty}(\tau_\infty, \ell)$ is a locally component of $\theta_D(\tau, 1)$ for a suitable D. (So $D_\infty = \mathrm{M}_\ell(\mathbb{H})$).

Archimedean case, V

- For decomposable φ, we have a decomposition

$$W_\varphi(1) = \lambda_\infty(\varphi_\infty) \cdot \lambda_{fin}(\varphi_{fin}).$$

- Assume that the dimension of models for $\theta_{D_\infty}(\tau_\infty, \ell)$ is greater than 1.
- The Kirillov model: there exists σ_∞ such that $\dim \sigma_\infty > 1$,

$$\mathcal{K}_\infty := \mathrm{ind}_{N_{n,D}}^{P_{n,D}} \sigma_\infty \ltimes \psi_{n,D} \hookrightarrow \theta_{D_\infty}(\tau_\infty, \ell).$$

- We choose a slice of the Kirillov model such that λ_∞ vanishes:

$$\tilde{\mathcal{K}}_\infty := \mathrm{ind}_{N_{n,D}}^{P_{n,D}} \psi_{n,D} \hookrightarrow \theta_{D_\infty}(\tau_\infty, \ell)$$

- Consider the $P_{n,D}(\mathbb{A})$-representation

$$\theta_D(\tau, \ell)_{fin} \otimes \tilde{\mathcal{K}}_\infty.$$

Then $W_\varphi(1) = 0$ for φ in this subspace. Contradiction.

Application

- In the construction of the twisted doubling integrals (joint with Friedberg, Ginzburg and Kaplan), it is important to use the generalized Speh representations $\theta(\tau, \ell)$ from a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$:
- This is a generalization of the doubling integrals of Piatetski-Shapiro and Rallis.
- This gives a family of Rankin-Selberg integrals for the tensor product L-functions for a classical group and a general linear group.
- To show that the global integral is Eulerian, we use the unique degenerate model of $\theta(\tau, \ell)$.

To extend the twisted doubling integrals to the case of quaternionic unitary groups, representations of $\mathrm{GL}_{n,D}$ with unique models are required. (Analogues of the Speh representations.)