$X / \mathbb{F}_q$, G - reductive group

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) = \mathrm{Autom}$$

<u>Thm</u> (V. Lafforgue)

$$\mathrm{Autom}_{cusp} = \bigoplus_\sigma \mathrm{Autom}_{cusp,\sigma}$$

σ - semi-simple Langlands parameter

$$\mathrm{Weil}(X) \longrightarrow \check{G}$$

<u>Thm</u> (C. Xue)

$$\mathrm{Autom} \curvearrowleft A_{/\overline{\mathbb{Q}}_\ell}$$

$\mathrm{Spec}(A)$ = semi-simple Langlands parameters.

$$\sigma \rightsquigarrow \mathcal{F}_\sigma \in \mathrm{Shv}(\mathrm{Bun}_G)$$

$$\mathrm{Funct}(\mathcal{F}_\sigma) \in \mathrm{Autom}_\sigma$$

---

k - ground field

$$\begin{array}{ccc} \cancel{\mathrm{QCoh}}(\mathrm{LocSys}_{\check{G}}(X)) & \xrightarrow{\mathbb{L}} & \text{D-mod}(\mathrm{Bun}_G) \\ \mathrm{IndCoh} \qquad \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ k_\sigma & \longrightarrow & \mathcal{F}_\sigma \end{array}$$

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}(X)) \overset{\mathbb{L}}{\simeq} \text{D-mod}(\mathrm{Pic})$$

$$\mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m}(X)} \longleftrightarrow \overset{\rotatebox{90}{$\in$}}{\delta_1}$$

$X/\mathbb{C}$ $\qquad$ e - field of coefficients

Shv^{all}

BZ-Nadler.

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \xrightarrow{\sim} \mathrm{Shv}^{all}_{Nilp}(\mathrm{Bun}_G)$$

$\mathcal{Y}$

$\mathcal{W} \subseteq T^*\mathcal{Y}$

$$\mathrm{Shv}^{all}_{\omega}(\mathcal{Y}) \subseteq \mathrm{Shv}^{all}(\mathcal{Y})$$
$$\mathrm{Shv}_{\omega}(\mathcal{Y}) \subseteq \mathrm{Shv}(\mathcal{Y})$$
$$\mathrm{Dmod}_{\omega}(\mathcal{Y}) \subseteq \mathrm{Dmod}(\mathcal{Y})$$

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}(X)) = \mathrm{Shv}^{all}_{loc.const}(\mathrm{Pic})$$

$$\mathrm{Nilp} \subseteq T^*\mathrm{Bun}_G = (\mathcal{M}, \mathcal{M} \xrightarrow{A} \mathcal{M} \otimes \omega_X)$$

Arinkin, Raskin, Kazhdan, Rozenblyum, Varshavsky

$$\mathrm{QCoh}(\mathrm{LocSys}^{restr}_{\check{G}}(X)) \overset{LL}{\simeq} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

$\mathrm{LocSys}^{restr}_{G}(X)$ is a new algebro geometric object that makes sense in any sheaf theory.

$$\mathrm{LocSys}^{restr}_{\check{G}}(X) \hookrightarrow \mathrm{LocSys}_{\mathrm{Betti}}$$

$$\begin{array}{ccc} \| & & \wr\| \text{ as st.} \\ \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) & \hookrightarrow & \mathrm{LocSys}_{dR} \end{array}$$

$$\begin{tikzcd} \widetilde{\mathrm{Pic}} \arrow[d, "\pi"] \\ \mathrm{Pic} \end{tikzcd}$$

$\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$

$$\begin{tikzcd} \mathrm{LocSys}_{\check{G},\mathrm{Betti}}(X) \arrow[d, "r"] \arrow[r, phantom, "="] & \mathrm{Hom}(\pi_1(X), \check{G})/\mathrm{Ad}\,\check{G} \arrow[d] \\ \mathrm{LocSys}^{\mathrm{coarse}}_{\check{G},\mathrm{Betti}}(X) \arrow[r, phantom, ":="] & \mathrm{Hom}(\pi_1(X), \check{G})/\!/\mathrm{Ad}\,\check{G} \end{tikzcd}$$

$$\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) := \bigsqcup_{\sigma \in \mathrm{LocSys}^{\mathrm{coarse}}_{\check{G},\mathrm{Betti}}(X)} \left(\mathrm{LocSys}_{\check{G},\mathrm{Betti}}(X)\right)^{\wedge}_{r^{-1}(\sigma)}$$

$\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is a pre-stack over the field of coefficients.

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) =$$

$$= \mathrm{Funct}^{\mathrm{sym.mon}}(\mathrm{Rep}(\check{G}), A\text{-mod} \otimes \mathrm{Lisse}(X))$$

A - com. algebra over the field of coefficients.

$$\mathrm{Lisse}(X) \subseteq \mathrm{Shv}(X)$$

$$\shortparallel$$

$$\mathrm{Ind}(\mathrm{Shv}(X)^{\mathrm{constr}})$$

<u>Example</u> $X = S^1$.

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{LocSys}_{\check{G},\mathrm{Betti}}(X)) =$$

$$= \mathrm{Funct}^{\mathrm{Sym\,Mon}}(\mathrm{Rep}(\check{G}), (A\text{-mod})^{\circlearrowleft})$$

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{LocSys}^{\mathrm{restr}(X)}_{\check{G},\mathrm{Betti}}) =$$

corresponds to the condition that our automorphism of the A-mod is locally finite over

A-local is locally ...
the field of coefficients.

$$\text{A-ad} \otimes \mathrm{Shv}^{\vee}_{\text{loc.const}}(S^1) = \text{A-ad}^{\curvearrowleft}$$

$$\cup\!\mid$$

$$\text{A-ad} \otimes \mathrm{Lisse}(X) = \begin{array}{l}\text{automorphism is}\\ \text{locally finite.}\end{array}$$

$$\begin{array}{c} \mathbb{R} \\ \big\downarrow{\scriptstyle \pi} \\ S^1 \end{array}$$

<u>**Properties**</u>: $\mathrm{LocSys}^{\mathrm{rest}}_{\check{G}}(X)$

$$\| $$

$$\mathrm{LocSys}^{\mathrm{rest},\mathrm{rigid}_x}_{\check{G}}(X) / \mathrm{Ad}(\check{G})$$

$\mathrm{LocSys}^{\mathrm{rest},\mathrm{rigid}_x}_{\check{G}}(X)$ = disjoint union of formal affine schemes.

$$\mathrm{Spec}(A)^{\wedge}_{Z} \qquad \underset{p}{Z} \subseteq \mathrm{Spec}(A)$$

finite type over the field of coefficients.

$$\mathrm{LocSys}^{\mathrm{rest}}_{\check{G}}(X) \subseteq \mathrm{LocSys}_{\check{G},\mathrm{Betti}}(X)$$
$$\subseteq \mathrm{LocSys}_{\check{G},\mathrm{dR}}(X)$$

<u>Thm</u> | $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{rest}}_{\check{G}}(X)) =$

= compatible collections of functors

$$\mathrm{Rep}(\check{G})^{\otimes I} \longrightarrow \mathrm{Lisse}(X)^{\otimes I}$$

$\forall\ I \in \mathrm{fSets}$.

X is over $\overline{\mathbb{F}_q}$, but defined over $\mathbb{F}_q$.

$$X \xrightarrow{\mathrm{Frob}_X} X$$

$\mathrm{Frob} : \mathrm{LocSys}^{\mathrm{rest}}_{\check{G}}(X) \circlearrowleft$

$$\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X) = \left(\mathrm{LocSys}^{\mathrm{rest}}_{\check{G}}(X)\right)^{\mathrm{Frob}}.$$

Thm 2 $\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X)$ is an algebraic stack.

Thm 1'

$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X)) =$

compatible collection of functors

$$\mathrm{Rep}(\check{G})^{\otimes I} \longrightarrow \{\mathrm{Lisse}(X^I) + \text{partial Frobenius}\}$$

Cor Lafforgue's shtuka cohomologies

from an object of $\longleftarrow$

$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X))$.

$$A = \Gamma(\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}(X)})$$

$\mathrm{Spec}(A) =$ s.s. ~~Gal~~ Weil representations.

$\cap$

C:

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes C \longrightarrow C \otimes \mathrm{Lisse}(X)^{\otimes I}$$

lisse Hecke action of $\mathrm{Rep}(\check{G})$ on C.

<u>Thm 1''</u> For a compactly generated C TFAE:

- Lisse Hecke action
- Action $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$

$\mathrm{Shv}(\mathrm{Bun}_G)$

$$\begin{array}{ccc} \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) & \longrightarrow & \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \\ & \searrow & \cup\!\! \mid \\ & & \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(X^I) \\ & & \cup\!\! \mid \\ & & \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Lisse}(X)^{\otimes I} \end{array}$$

$$\mathrm{Shv}_{T^*\mathrm{Bun}_G \times \{0\}}(\mathrm{Bun}_G \times X^I) \subseteq \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$$

$$\| \qquad \leftarrow \qquad X\text{-proper.}$$

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Lisse}(X)^{\otimes I}$$

<u>Thm</u> (Nadler–Yun)

$$\begin{array}{c} \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\ \downarrow \\ \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X^I). \end{array}$$

<u>Cor</u> $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))$$

Cor $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) =$

$$= \bigoplus_{\sigma} \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)_{\sigma}.$$

Conj (GLC)

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{LL}{\simeq} \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$$

$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$ Autom. vector space.

X over $\overline{\mathbb{F}_q}$, but defined over $\mathbb{F}_q$.

$$\mathrm{Bun}_G \xrightarrow{\mathrm{Frob}_{\mathrm{Bun}_G}} \mathrm{Bun}_G.$$

$\mathbf{C}$ - compactly generated category

$$\Phi : \mathbf{C} \longrightarrow \mathbf{C}$$

$$\mathrm{Tr}(\Phi, C) \in \bar{V}\mathrm{ect}_e$$

$$c \in C$$

$$\begin{tikzcd} c \arrow[r, "\alpha"] \arrow[d, squiggly] & \Phi(c) \\ \mathrm{cl}(c, \alpha) \in \mathrm{Tr}(\Phi, C) & \end{tikzcd}$$

$$c_0 \xrightarrow{\beta} c_1$$

$$c_1 \xrightarrow{\gamma} \Phi(c_0)$$

$$c_0 \xrightarrow{\beta} c_1 \xrightarrow{\gamma} \Phi(c_0)$$

$$\mathrm{cl}(c_0, \gamma \circ \beta) \in \mathrm{Tr}(\Phi, C)$$

$$\parallel$$

$$\mathrm{cl}(c_1, \Phi(\beta) \circ \gamma)$$

$$c_1 \xrightarrow{\gamma} \Phi(c_0) \xrightarrow{\Phi(\beta)} \Phi(c_1)$$

$\mathcal{Y}$ - stack over $\overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$.

$$C = \mathrm{Shv}(\mathcal{Y})$$

$$\Phi = (\mathrm{Frob}_{\mathcal{Y}})_*$$

$$\mathcal{F} \longrightarrow (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F})$$

$$\Updownarrow$$

$$\mathrm{Frob}_{\mathcal{Y}}^*(\mathcal{F}) \longrightarrow \mathcal{F}.$$

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \xrightarrow{\sim} \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

$$\mathcal{Y} = \mathrm{Bun}_G$$

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \xrightarrow{\sim} \mathrm{Autom}$$

$$\mathrm{Tr}\left((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\right)$$

Then the composite map is an isomorphism.

<u>Cor 1</u>

$$\mathrm{Autom} = \mathrm{Tr}\left((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\right).$$

$$\mathrm{Drinf} \in \mathrm{QCoh}\left(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)\right)$$

<u>Conj 1'</u>

$$\mathrm{Drinf} \simeq \{\text{Lafforgue's object}\}$$

$$\mathrm{Autom} = \Gamma\left(\mathrm{LocSys}_{\check{G}}^{\mathrm{arith}}, \mathrm{Drinf}\right).$$

$$\mathrm{Drinf} = \omega_{\mathrm{LocSys}_{\check{G}}^{\mathrm{arith}}}.$$

Conj 2

$$A_{utom} = \Gamma\left(\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}, \omega_{\mathrm{LocSys}^{\mathrm{arith}}_{\check{G}}}\right).$$

$$V \in \mathrm{Rep}(\check{G})^{\otimes I} \rightsquigarrow$$

$$\mathrm{Sht}_V \in \mathrm{Lisse}(X^I)/\text{partial Frob.}$$
$$\in \mathrm{Shv}(X^I)$$

$$\begin{tikzcd}
\mathrm{Sht}_{X^I} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{H}_{X^I} \arrow[r, "s"] \arrow[d] & X^I \\
\mathrm{Bun}_G \arrow[r, "\mathrm{Id},\, \mathrm{Frob}_{\mathrm{Bun}}"'] & \mathrm{Bun}_G \times \mathrm{Bun}_G &
\end{tikzcd}$$

$$V \longrightarrow \mathrm{Sat}(V) \in \mathrm{Shv}(\mathcal{H}_{X^I})$$

$\mathrm{Sht}_V = (g \circ f)_! \, f^*(\mathrm{Sat}(V))$.

Conj 2' $\quad x^I \in X^I$

$$\left(\mathrm{Sht}_{X^I}\right)_{x^I} = \Gamma\left(\mathrm{LocSys}^{\mathrm{arth}}_{\check{G}}(X), \omega_{\mathrm{LocSys}} \otimes \mathcal{E}^V_{x_I}\right)$$

$$\mathcal{E}^V_{x_I} \in \mathrm{QCoh}\left(\mathrm{LocSys}^{\mathrm{arth}}_{\check{G}}(X)\right)$$

$\mathcal{Y}$ - q.q. stack.

$$\mathrm{Sing}_2(\mathcal{Y}) = \bigsqcup_y H^{-2}(T^*_y \mathcal{Y})$$

$\mathrm{Drinf} \in \mathbb{D}^{\Rightarrow}(\mathrm{Sing}_2(\mathcal{Y}))$.

$$\mathrm{Sing}_2(\mathrm{LocSys}^{\mathrm{arth}}_{\check{G}}(X)) = \left(\sigma;\ A \atop {\cap \atop H^0(X, \check{g}_\sigma)^{\mathrm{Frob}=q}}\right)$$

Γ