Construction of Euler Systems for $GSp_4 \times_{GL_4} GL_2$

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§1. Intro.

$\mathrm{Gal}_{\mathbb{Q}} \curvearrowright V$: p-adic rep., unram. outside $\Sigma \not\ni p$

$\forall \ell \notin \Sigma \cup \{p\}$, $P_\ell(X) = \det(I - X \cdot \mathrm{Frob}_\ell^{-1} \mid V)$

$T^*(1) \subset V^*(1)$: $\mathrm{Gal}_{\mathbb{Q}}$-stable lattice

<u>Def</u>. An ES for $(T^*(1), \Sigma)$ is $(c_n)_n$, $c_n \in H^1(\mathbb{Q}(\mu_n), T^*(1))$

$n = Mp^m$, M: sq-free prod. of $\ell \notin \Sigma \cup \{p\}$

such that

(wild) $\quad \mathrm{Nm}^{\mathbb{Q}(\mu_{Mp^{m+1}})}_{\mathbb{Q}(\mu_{Mp^m})} c_{Mp^{m+1}} = c_{Mp^m}$

(tame) $\quad \mathrm{Nm}^{\mathbb{Q}(\mu_{\ell Mp^m})}_{\mathbb{Q}(\mu_{Mp^m})} c_{\ell Mp^m} = P_\ell(\mathrm{Frob}_\ell^{-1}) \cdot c_{Mp^m}$, $\quad P_\ell(\mathrm{Frob}_\ell^{-1}) \in \mathrm{Gal}(\mathbb{Q}(\mu_{Mp^m})/\mathbb{Q})$

<u>App</u> Bound Selmer groups & Bloch-Kato conj.s

<u>Thm</u> (H-Jin-Sakamoto)

$\pi = \sigma \otimes \tau$: cusp auto rep for $GSp_4 \times_{GL_4} GL_2 =: G$

unram. outside $\Sigma \not\ni p$

σ_∞: disc. series for GSp_4 of wt (k_1, k_2)

τ_∞: $\quad GL_2 \quad k_3$

$\rightsquigarrow V = V_\pi$: 8-dim $\mathrm{Gal}_{\mathbb{Q}}$-rep.

s.t. $\forall \ell \notin \Sigma \cup \{p\}$, $P_\ell(\ell^{-s}) = L_\ell\left(s - \frac{k_1+k_2+k_3-4}{2}, \pi\right)$

w.r.t. ${}^L G = G(\mathbb{C}) \hookrightarrow GL_8(\mathbb{C})$

Assume π is non-endoscopic, generic, p-ordinary

Then $\exists$ ES for $(T^*(1), \Sigma)$ for some $T^*(1) \subset V^*(1)$

Rmk. Idea follows Loeffler - Skinner - Zerbes for GSp_4

✓ ① construction

② integrality : extensive computation

③ wild : norm-compatibility machine (Loeffler)

✓ ④ tame : local rep. theory

§2. Integral formula for L-function

$$H = GL_2 \underset{GL_1}{\times} GL_2 \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$\downarrow \iota \qquad\qquad \downarrow$$

$$G = GSp_4 \underset{GL_1}{\times} GL_2 \qquad \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

For $\varphi_\pi \in \pi$, $\phi : \mathbb{A}_f^2 \to \mathbb{C}$ Schwartz function

consider

$$(*)\quad L(\varphi_\pi, \phi, s) := \int_{\underset{\substack{\uparrow \\ \text{scalar matrices}}}{C}(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi_\pi(h)\, \mathrm{Eis}^{\phi}(h_1, \underset{\substack{\uparrow \\ \text{central char} \\ \text{of } \pi}}{\omega}, s)\, |\det h_1|\, dh$$

π is generic $\rightsquigarrow$ unfold $\mathrm{Eis}^{\phi}(h_1, \omega, s) = \sum_{(B\backslash GL_2)(\mathbb{Q})} f^{\phi}_{\omega, s}(h_1)$

where $f^{\phi}_{\omega,s}(h_1) = \int_{\mathbb{A}^\times} \phi((0,t)h_1)\,\omega(t)\,|t|^{2s}\, d^\times t$

Get $$L(\varphi_\pi, \phi, s) = \int_{(C\underset{\substack{\uparrow \\ \left(\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right)}}{N}\backslash H)(\mathbb{A})} W_{\varphi_\pi}(h)\, f^{\phi}_{\omega,s}(h_1)\, |\det h_1|\, dh$$

$$= \prod_\ell L_\ell(\varphi_{\pi_\ell}, \phi_\ell, s)$$

Then $L_\ell(s, \pi)$ is rep by $L_\ell(\varphi_{\pi_\ell}, \phi_\ell, s)$.

§3. Construction of c_n

Y_H, Y_G : Sh. var. for H, G

$$\begin{array}{ccc} \underset{\substack{\circlearrowleft \\ \mathcal{H}(H)}}{H^1_{et}}(Y_H, \underset{\substack{\Downarrow \\ \mathrm{Eis}\,\phi}}{\mathrm{Sym}^k(1) \boxtimes 1}) & \xrightarrow{\iota_*} & \underset{\substack{\circlearrowleft \\ \mathcal{H}(G)}}{} H^5_{et}(Y_G, \mathcal{L}) \\ & & \big\downarrow \mathrm{pr}_{\pi^\vee} \left(\begin{array}{c}\text{Hochschild} \\ \text{- Serre}\end{array}\right) \\ & & H^1(\mathbb{Q}, V^*_\pi(1-n)) \end{array}$$

$$\rightsquigarrow \mathrm{Symbl} : H^1_{et}(Y_H, \mathrm{Sym}^k(1)\boxtimes 1) \otimes_{\mathcal{H}(H)} \mathcal{H}(G) \to H^5_{et}(Y_G, \mathcal{L})$$

Will construct $c_n = pr_{\pi^\vee} \circ \mathrm{Symbl}(EB_{\underline{\underline{\phi_n}}} \otimes \xi_n)$

$\phi_n = ch((0,1) + n\hat{\mathbb{Z}}^2)$

eg. $k=0$. $EB_{\phi_n} = g_{0,\frac{1}{n}}$

Siegel units

ξ_n is chosen to make norm relation hold.

§4. Choice of ξ_n

$$H^1_{et}(Y_H, \mathrm{Sym}^k(1)\boxtimes 1) \supset \{EB_\phi\} \cong \bigoplus I(\chi, \psi)$$

$\chi = |\cdot|^{k+\frac{1}{2}} \cdot \nu$ (fin. order), $\psi = |\cdot|^{-1/2}$

$GL_2(\mathbb{A}_f)$-equivariant

$$\mathrm{Symbl} = \bigotimes_\ell \mathrm{Symbl}_\ell$$

To compute with Symbl_ℓ, it suffices to understand

$$I(\chi,\psi) \otimes_{\mathcal{H}(H)} \mathcal{H}(G) \to \pi^\vee$$

$$\underset{\text{Frob. reciprocity}}{\longleftrightarrow} \quad I(\chi,\psi) \otimes \pi\big|_H \to \mathbb{C}.$$

There is at most 1-dim of such homo.,

a conseq of GGP conj. for $SO_4 \hookrightarrow SO_5$

$$\begin{array}{ccc} SO_4 & \hookrightarrow & SO_5 \\ \wr\| & & \wr\| \\ H/C & & GSp_4/C \end{array}$$

Construct a non-zero such homo which allows explicit computation

($\rightsquigarrow$ choice of $\xi_n = \otimes \xi_\ell$)

Make use of

$$L_\ell(\varphi_\pi, \phi, s) = \int_{(CN\backslash H)(\mathbb{Q}_\ell)} W_{\varphi_\pi}(h) f^{\phi}_{\omega,s}(h_1) |\det h_1| \, dh$$

$\nearrow \varphi_\pi = \varphi_\sigma \otimes \varphi_\tau$

a finite sum of

$$\int_{\mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times} W_{\varphi_\sigma}\begin{pmatrix} xy & & & \\ & x & & \\ & & y & \\ & & & 1 \end{pmatrix} W_{\varphi_\tau}\begin{pmatrix} x & \\ & y \end{pmatrix} |x|^{s-2} |y|^s \, d^\times x \, d^\times y$$

$$\parallel$$

$$Z(\varphi_\pi, s)$$

By Casselman-Shalika formula,

$$Z(\underset{\substack{\uparrow \\ \text{normalized spherical vector}}}{\varphi^\circ_\pi}, s) = L_\ell(s, \pi) \cdot L_\ell(2s, \underset{\substack{\uparrow \\ \text{central char.}}}{\omega})^{-1}$$

$\underline{\text{Thm.}}$ $\exists\ \xi_s(\ell) \in \mathcal{H}_\ell(G)$ s.t. $Z(\xi_s(\ell)\cdot\varphi_\pi^0, s) = 1.$

(This is the desired $\xi_\ell = \xi_0(\ell)$).

Let $z_{\varphi,s} : H(\mathbb{Q}_\ell) \to \mathbb{C}$

$$h \mapsto Z(h\cdot\varphi, s)$$

Then $z_{\varphi,s} \in I(\psi^{-1}, \chi^{-1}) \otimes I(|\cdot|^{1/2}, |\cdot|^{-1/2})$.

where $\chi = \omega^{-1}|\cdot|^{1/2}$, $\psi = |\cdot|^{-1/2}$

and $\begin{cases} z_{\varphi_\pi^0,s}(1) = L_\ell(s,\pi)\cdot L_\ell(2s,\omega)^{-1} \\ z_{\xi_s(\ell)\varphi_\pi^0,s}(1) = 1 \end{cases}$

Define $\mathfrak{z}_{\chi,\psi} : I(\chi,\psi)\otimes\pi|_H \to \mathbb{C}$

by $\mathfrak{z}_{\chi,\psi}(f\otimes\varphi) = \langle M(f)\otimes 1, z_{\varphi,s}\rangle$

where $M : I(\chi,\psi)\to I(\psi,\chi)$ intertwining operator

$\langle\ ,\ \rangle : I(\psi,\chi)\otimes I(\psi^{-1},\chi^{-1}) \to \mathbb{C}$

And

$$\mathfrak{z}_{\chi,\psi}(F_{\phi_1}^{\chi,\psi}\otimes\xi_0(\ell)\varphi_0) = \frac{1}{\ell+1}L(0,\pi)^{-1}\mathfrak{z}_{\chi,\psi}(F_{\phi_0}^{\chi,\psi}\otimes\varphi_0)$$

Here $L_\ell(2s, \omega)^{-1}$ disappears because

$$M(F_{\phi_1}^{\chi,\psi})(1) = L_\ell\left(1, \tfrac{\psi}{\chi}\right) L_\ell\left(1, \tfrac{\chi}{\psi}\right)^{-1}$$

$$= L_\ell(1, \omega|\cdot|^{-1}) = L_\ell(0, \omega)$$

$$M(F_{\phi_0}^{\chi,\psi})(1) = L_\ell\left(1, \tfrac{\chi}{\psi}\right)^{-1}$$

Rmk. $\zeta_s(\ell) = 1 - \frac{1}{\ell^{s+2}} U_1(\ell) + \frac{2}{\ell^{2s+3}} U_2(\ell) + \frac{1}{\ell^{2s+3}} U_3(\ell)$

$$+ \frac{1}{\ell^{2s+3}} U_4(\ell) + \frac{1}{\ell^{3s+4}} U_5(\ell) + \frac{1}{\ell^{4s+4}} U_6(\ell)$$

w.r.t. $\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix}, \begin{pmatrix} * & * \\ * & * \end{pmatrix}$

$U_1(\ell)$: $\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell & \\ & 1 \end{pmatrix}$

$U_2(\ell)$: $\begin{pmatrix} \ell^2 & & & \\ & \ell & & \\ & & \ell & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell & \\ & \ell \end{pmatrix}$

$U_3(\ell)$: $\begin{pmatrix} \ell^2 & & & \\ & \ell & & \\ & & \ell & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell^2 & \\ & 1 \end{pmatrix}$

$U_4(\ell)$: $\begin{pmatrix} \ell^2 & & & \\ & \ell^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell & \\ & \ell \end{pmatrix}$

$U_5(\ell)$: $\begin{pmatrix} \ell^3 & & & \\ & \ell^2 & & \\ & & \ell & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell^2 & \\ & \ell \end{pmatrix}$

$U_6(\ell)$: $\begin{pmatrix} \ell^4 & & & \\ & \ell^2 & & \\ & & \ell^2 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ell^2 & \\ & \ell^2 \end{pmatrix}$

$\zeta_s(\ell)$ was found by explicitly computing with

$Z(v_i(\ell)\varphi^{\circ}_{\pi}, s)$ using Casselman–Shalika

Would like a general philosophy of why it works