A Local Trace Formula for the Local Gan-Gross-Prasad Conjecture for Special Orthogonal Groups

Zhilin Luo

University of Minnesota

Automorphic forms and arithmetic, Columbia university
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$$L^2(S^2) \simeq \widehat{\bigoplus}_{l=0}^{\infty} H_l,$$

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- $$m(\pi) = \frac{\int_{\mathrm{SO}_2(\mathbb{R})} \Theta_\pi(h)dh}{\mathrm{vol}(\mathrm{SO}_2(\mathbb{R}), dh)}, \quad \text{by Schur's orthogonality.}$$

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- $\xi =$ a generic character of N extending to H.
- (G, H, ξ) is called a **Gan-Gross-Prasad** triple.

Multiplicity one

- Set

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \xi_F), \quad \pi \in \mathrm{Irr}(G(F))$$

Theorem.

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- For F Archimedean, proved by B. Sun-C. Zhu for $r = 0$, and D. Jiang-Sun-Zhu reducing the general case to $r = 0$.

Local Gan-Gross-Prasad conjecture

- The local Gan-Gross-Prasad conjecture suggests that $m(\pi)$ has more stable behavior by considering the local **Vogan packet** attached to (G, H, ξ).

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- To introduce local Vogan packets, consider pure inner forms of $\mathrm{SO}(W)$, parametrized by $H^1(F, \mathrm{SO}(W)) \simeq H^1(F, H)$
- For $\alpha \in H^1(F, H)$, there exists

$$(W_\alpha, V_\alpha = W_\alpha \oplus W^\perp)$$

dim $W_\alpha =$ dim W, $\mathrm{disc} W_\alpha = \mathrm{disc} W$,
with a GGP triple

$$(G_\alpha, H_\alpha, \xi_\alpha).$$

Moreover

$${}^L G_\alpha \simeq {}^L G.$$

Local Gan-Gross-Prasad conjecture

Conjecture.(Gan-Gross-Prasad)
For any generic L-parameter $\varphi : \mathcal{W}_F \to {}^L G$ with L-packet $\Pi^G(\varphi)$,

$$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi) = 1.$$

Moreover, the non-vanishing of $m(\pi)$ is detected by representations of the component group A_φ attached to φ, which is related to the sign of the relevant local symplectic root numbers.

- $$\varphi \text{ is } \begin{cases} \text{generic}, & L(s, \varphi, \mathrm{Ad}) \text{ is holomorphic at } s = 1 \\ \text{tempered}, & \mathrm{Im}(\varphi) \text{ is bounded} \end{cases}$$

Local Gan-Gross-Prasad conjecture: p-adic

- J.-L. Waldspurger (tempered) and C. Moeglin-Waldspurger (generic) proved the conjecture completely when F is p-adic (Assuming LLC for non quasi-split SO and quasi-split SO_{2n}).

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- There are parallel conjectures for skew-hermitian unitary groups and symplectic-metaplectic groups.
  Gan-Ichino proved the conjecture for skew-hermitian unitary groups, and H. Atobe for symplectic-metaplectic groups, via theta correspondence when F is p-adic.

Local Gan-Gross-Prasad conjecture: Archimedean

- For unitary groups, when $F = \mathbb{R}$,
  Beuzart-Plessis proved the multiplicity part of the conjecture for φ tempered.
  H. He proved the conjecture for discrete series representations.
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- For special orthogonal groups, when $F = \mathbb{C}$,
  J. Möllers proved the conjecture for $\mathrm{SO}(n) \times \mathrm{SO}(n+1)$.

The theorem

In the special orthogonal groups setting, we prove the following theorem.

Theorem (L.)

For any tempered L-parameter $\varphi : \mathcal{W}_F \to {}^L G$,

$$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi) = 1.$$

- We follow the approach of Waldspurger and Beuzart-Plessis.

Local trace formula

- For $\pi \in \mathrm{Temp}(G(F))$, by Frobenius reciprocity for unitary representations,

$$\mathrm{Hom}_{H(F)}(\pi, \xi_F) \simeq \mathrm{Hom}_{G(F)}(\pi, \mathrm{Ind}_H^G \xi_F)$$

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- $$L^2(H(F)\backslash G(F), \xi_F) \curvearrowleft G(F) \quad \text{spectral decomposition.}$$

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- Following Arthur,

$$L^2(H(F)\backslash G(F), \xi) \curvearrowleft \mathcal{C}_c^\infty(G(F)) \quad \text{via convolution.}$$

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- For $f \in \mathcal{C}_c^\infty(G(F))$, $x \in G(F)$, $\varphi \in L^2(H(F)\backslash G(F),\xi_F)$,

$$(R(f)\varphi)(x) = \int_{G(F)} f(g)\varphi(xg)dg = \int_{H(F)\backslash G(F)} K_f(x,y)\varphi(y)dy$$

  where

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- $R(f)$ has an integral kernel $K_f(x,y)$.

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$$\mathrm{Tr}(R(f)) \sim \int_{H(F)\backslash G(F)} K(x,x)dx.$$

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- $f \in \mathcal{C}_c^\infty(G(F))$ is called strongly cuspidal if
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for any proper parabolic subgroup $P = MU$ of G.
- Similarly, define strongly cuspidal functions in the *Harish-Chandra Schwartz space* $\mathcal{C}(G(F))$ of $G(F)$, denoted as $\mathcal{C}_{\mathrm{scusp}}(G(F))$.

Local trace formula

Theorem (L.)

For $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$,

$$J(f) = \int_{H(F)\backslash G(F)} K_f(x,x)dx$$

is absolutely convergent.

- Establish spectral and geometric expansions for $J(f)$ through comparing with Arthur's local trace formula.

Spectral expansion

Theorem (L.)

For $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$, *set*

$$J_{\mathrm{spec}}(f) = \int_{\mathcal{X}(G(F))} D(\pi)\theta_f(\pi)m(\pi)d\pi.$$

Then $J_{\mathrm{spec}}(f)$ *is absolutely convergent, and*

$$J(f) = J_{\mathrm{spec}}(f).$$

- $\mathcal{X}(G(F)) := \{(M,\sigma) | \quad \sigma \in T_{\mathrm{ell}}(M(F))\}/\mathrm{conj.}$, where $T_{\mathrm{ell}}(M(F)) =$ elliptic representations introduced by Arthur.

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- For π attached to (M,σ), $\theta_f(\pi) = (-1)^{a_G - a_M} J_M^G(\sigma, f)$, where $J_M^G(\sigma, f)$ is the weighted character defined by Arthur.

Spectral expansion

- Introduce $\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ with

$$\mathcal{L}_\pi \neq 0 \Leftrightarrow m(\pi) \neq 0, \quad \pi \in \mathrm{Temp}(G(F)).$$

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- For $\pi \in \mathrm{Temp}(G(F))$, set

$$\mathcal{L}_\pi(T) = \int_{H(F)}^{*} \mathrm{Tr}(\pi(h^{-1})T)\xi_F(h)dh, \quad T \in \mathrm{End}(\pi)^\infty.$$

In general, the integral is not absolutely convergent, need regularization. (Waldspurger, Lapid-Mao, Sakellaridis-Venkatesh, Beuzart-Plessis).

Spectral expansion

- Insert $\mathcal{L}_\pi$ into the Plancherel formula on $G(F)$. More precisely,

$$\begin{aligned} K(f,x) &= \int_{H(F)} f(x^{-1}hx)dh \\ &= \int_{\mathcal{X}_{\mathrm{temp}}(G(F))} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))d\pi. \end{aligned}$$

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- By continuity, assume $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$ has compactly supported Plancherel transform.
- By compactness, choose $f' \in \mathcal{C}(G(F))$ such that

$$\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} = m(\pi)$$

for any $\pi \in \mathcal{X}_{\mathrm{temp}}(G(F))$ with $\pi(f) \neq 0$.

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Spectral expansion

- Compare the resulting formula with the spectral side of Arthur's local trace formula.
- $$K(f,x) = \int_{\mathcal{X}_{\mathrm{temp}}(G(F))} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi.$$
  $$= \int_{H(F)} \xi_F(h)dh \int_{H(F)} \xi_F(h')dh' \int_{G(F)} f(x^{-1}hgh'x)f'(g)dg.$$
- Therefore
  $$J(f) = \int_{H(F)\backslash G(F)} dx \int_{H(F)} \xi(h)dh$$
  $$\int_{H(F)} \xi(h')dh' \int_{G(F)} f(x^{-1}hgh'x)f'(g)dg$$

Spectral expansion: comparasion with Arthur's trace formula

- After introducing truncation, showing the integral order can be switched, and changing variables

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- $J(f)$ is equal to

$$J(f) = \int_{\mathcal{X}(G(F))} D(\pi)\theta_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi$$
$$= \int_{\mathcal{X}(G(F))} D(\pi)\theta_f(\pi)m(\pi)d\pi,$$

Geometric multiplicity formula

Theorem (L.)

For $\pi \in \mathrm{Temp}(G(F))$,

$$m(\pi) = m_{\mathrm{geom}}(\pi) = \int_{\Gamma(G,H)} c_\pi(x) D^G(x)^{1/2} \Delta(x)^{-1/2} dx.$$

- When F is p-adic it was proved by Waldspurger.

Geometric multiplicity formula: $\Gamma(G, H)$

- $$\Gamma(G,H) := \bigcup_{T \in \mathcal{T}} T_{\mathrm{reg}}(F).$$

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  $\mathcal{T}$ is a set of subtori of $\mathrm{SO}(W)$.

- $T \in \mathcal{T}$ iff. T max. ell. in $\mathrm{SO}(W'')$
  where $W'' \subset W$ non-degenerate and $\dim(W/W'')$ even.

Geometric multiplicity formula: definition of c_π

Theorem (Harish-Chandra for p-adic, Barbasch-Vogan for Archimedean)

For $x \in G_{\mathrm{ss}}$ and $X \in \omega \subset \mathfrak{g}_x$ a small neighborhood of 0, there exists constants $c_{\pi,\mathcal{O}}(x) \in \mathbb{C}$ such that

$$\lim_{X \to 0} D^G(xe^X)^{1/2}\Theta_\pi(xe^X) = D^G(x)^{1/2} \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, X).$$

Here $\widehat{j}(\mathcal{O}, X) = \mathcal{F}(J_{\mathcal{O}}(\cdot))$.

- The definition of c_π, first appeared in the work of Waldspurger, is the main technical ingredient.

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- The definition of c_π, first appeared in the work of Waldspurger, is the main technical ingredient.
- c_π is nonzero only when G_x is quasi-split. When it is the case, $c_\pi = c_{\pi,\mathcal{O}}$ for a particular $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.

Geometric multiplicity formula: definition of c_π

- For unitary groups, $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$ can be permuted by scaling. The geometric multiplicity is independent of the orbit chosen. Therefore set

$$c_\pi(x) := \frac{\sum_{\mathcal{O}\in\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|}.$$

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$$c_\pi(x) := \frac{\sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\pi,\mathcal{O}}}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|}.$$

- Benefit:

$$D^G(x)^{1/2} c_\pi(x) = \lim_{x'\in T_{\mathrm{qd},x}(F)\to x} \frac{D^G(x')\Theta_\pi(x')}{|W(G_x, T_{\mathrm{qd},x})|}$$

where $T_{\mathrm{qd},x} \subset B_x \subset G_x$.

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- It is NOT the case for special orthogonal groups, really need to pick up a particular regular nilpotent orbit.

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- $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V)) \neq \emptyset$ iff. (V, q) is quasi-split.
  For $\dim V$ is odd or ≤ 2, $|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V))| = 1$.

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- For dim $V = 2m$ is even and ≥ 4, set

$$\mathcal{N}^V = \begin{cases} F^\times/F^{\times 2}, & \text{split} \\ \mathrm{Im}(q_{\mathrm{an}})/F^{\times 2}, & \text{non-split.} \end{cases}$$

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  Then $\mathcal{N}^V \leftrightarrow \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{so}(V))$.
- Therefore

$$\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}) \leftrightarrow \begin{cases} \mathcal{N}^V, & \dim V \text{ is even } \geq 4, \\ \mathcal{N}^W, & \dim W \text{ is even } \geq 4. \end{cases}$$

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with $G'_x = \mathrm{SO}(V'_x) \times \mathrm{SO}(W'_x)$, $G''_x = T \times T$.
- When G'_x is quasi-split, set

$$c_\pi(x) = \begin{cases} c_{\pi,\mathcal{O}_{\nu_0}}, & \dim V'_x \geq 4 \text{ even} \\ c_{\pi,\mathcal{O}_{-\nu_0}}, & \dim W'_x \geq 4 \text{ even} \\ c_{\pi,\mathcal{O}_{\mathrm{reg}}}, & \text{otherwise.} \end{cases}$$

The proof

The following properties are needed for φ a tempered L-parameter.

STAB For any $\alpha \in H^1(F, H)$,

$$\Theta_{\alpha,\varphi} = \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} \Theta_\pi.$$

is stable.

TRANS For $\alpha \in H^1(F, H)$, $\Theta_{\alpha,\varphi}$ is the transfer of $e(G_\alpha)\Theta_\varphi$, where $e(G_\alpha) \in Br_2(F)$ is the Kottwitz sign. $Br_2(F) = \{\pm 1\}$ if $F \neq \mathbb{C}$.

WHITT For G quasi-split and every $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, there exists a unique representation in $\Pi^G(\varphi)$ admitting a Whittaker model of type $\mathcal{O}$.

The proof

- For F Archimedean, LLC is known by R. Langlands, [STAB] and [TRANS] is known by D. Shelstad, and [WHITT] follows from B. Kostant and D. Vogan.

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- For F p-adic, LLC is known from Arthur for quasi-split special orthogonal groups (need refinment for SO_{2n}).
- For non quasi-split special orthogonal groups it is expected to follow from the last chapter of Arthur's book.

The proof

Lemma (L.)

For any $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, *define*

$$c_{\varphi,\mathcal{O}}(x) := \sum_{\pi \in \Pi^G(\varphi)} c_{\pi,\mathcal{O}}(x).$$

Then

$$c_{\varphi,\mathcal{O}}(x) = c_{\varphi,\mathcal{O}'}(x)$$

for any $\mathcal{O}, \mathcal{O}' \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$.
In particular,

$$D^G(x)^{1/2} c_{\varphi,\mathcal{O}}(x) = |W(G_x, T_{\mathrm{qd},x})|^{-1} \lim_{x' \in T_{\mathrm{qd},x}(F) \to x} D^G(x') \sum_{\pi \in \Pi^G(\varphi)} \Theta_\pi(x').$$

The proof

- $$\sum_{\alpha \in H^1(F,H)} \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} m(\pi) =$$
$$\int_{\Gamma_{\mathrm{stab}}(G,H)} c_\varphi(x) \left\{ \sum_{\alpha \in H^1(F,H)} \sum_{y \in \Gamma(G_\alpha, H_\alpha), y \sim_{\mathrm{stab}} x} e(G_\alpha) \right\} dx.$$

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- $$\sum_{\alpha \in H^1(F,H)} \sum_{y \in \Gamma(G_\alpha, H_\alpha), y \sim_{\mathrm{stab}} x} e(G_\alpha) = 0$$
unless $x = 1$.

The proof

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unless $x = 1$.

- $$\sum_{\alpha\in H^1(F,H)} \sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi) = c_\varphi(1) = 1.$$
where the last identity follows from F. Rodier when F is p-adic, and H. Matumoto when F is Archimedean.

Geometric expansion

Theorem (L.)

For $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$, *set*

$$J_{\mathrm{geom}}(f) = \int_{\Gamma(G,H)} c_f(x) D^G(x)^{1/2} \Delta(x)^{-1/2} dx.$$

Then $J_{\mathrm{geom}}(f)$ *is absolutely convergent, and*

$$J(f) = J_{\mathrm{geom}}(f).$$

Geometric expansion: definitions

- Set
$$\theta_f(x) = (-1)^{a_G - a_{M(x)}} D^G(x)^{-1/2} J^G_{M(x)}(x, f).$$
Then $\theta_f(x)$ is conjugation invariant.

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Then $\theta_f(x)$ is conjugation invariant.
- It is a **quasi-character**, i.e.
$$\lim_{X \to 0} D^G(xe^X)^{1/2}\theta_f(xe^X) = D^G(x)^{1/2} \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\theta_f, \mathcal{O}}(x) \widehat{j}(\mathcal{O}, X).$$

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- Define
$$c_f(x) = \begin{cases} c_{\theta_f,\mathcal{O}_{\nu_0}}, & \dim V'_x \geq 4 \text{ even} \\ c_{\theta_f,\mathcal{O}_{-\nu_0}}, & \dim W'_x \geq 4 \text{ even} \\ c_{\theta_f,\mathcal{O}_{\mathrm{reg}}}, & \text{otherwise.} \end{cases}$$

Geometric expansion: localization

- By partition of unity,

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- For $x \in \mathrm{SO}(W)_{\mathrm{ss}}$, when $x \neq 1$,

$$(G_x, H_x, \xi_x) = (G'_x, H'_x, \xi'_x) \times (G''_x, H''_x, 1).$$

  (G'_x, H'_x, ξ'_x) is a GGP triple of smaller dimension, and $(G''_x, H''_x, 1)$ is $\Delta : H''_x \hookrightarrow H''_x \times H''_x = G''_x$.

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- Induction on dim G and Arthur's local trace formula.

Geometric expansion: Lie algebra variant

- For supp $\theta_f \subset$ neighborhood of $x = 1$, via exponential, descent to Lie algebra variants $J_{\mathrm{geom}}^{\mathrm{Lie}}(f)$ and $J^{\mathrm{Lie}}(f)$.

Geometric expansion: Lie algebra variant

- For supp $\theta_f \subset$ neighborhood of $x = 1$, via exponential, descent to Lie algebra variants $J^{\mathrm{Lie}}_{\mathrm{geom}}(f)$ and $J^{\mathrm{Lie}}(f)$.
- $J_{\mathrm{geom}}(f)$ contains asymptotic of weighted orbital integrals near singular locus, but Arthur's local trace formula only has regular semi-simple locus. Cannot compare directly.

Geometric expansion: Lie algebra variant

- Perform a Fourier transform on $\mathfrak{h} = \mathrm{Lie} H$ to resolve the possible singularities,

$$K^{\mathrm{Lie}}(f, x) = \int_{\mathfrak{h}} f(gXg^{-1})\xi_F(X)dX = \int_{\Xi+\mathfrak{h}^\perp} \widehat{f}(g^{-1}Xg)dX.$$

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- $$J^{\mathrm{Lie}}(f) = \int_{H(F)\backslash G(F)} dg \int_{\Xi+\mathfrak{h}^{\perp}} \widehat{f}(g^{-1}Xg)dX.$$

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- $\Gamma(\Xi + \mathfrak{h}^{\perp}) = G(F)$-conjugacy classes of regular semi-simple elements in $\Xi + \mathfrak{h}^{\perp}$.

Geometric expansion: Lie algebra variant

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- $\widehat{j}(X,\cdot) = \mathcal{F}(J(X,\cdot))$ and

$$\lim_{t\in F^{\times 2}, t\to 0} D^G(X,tY)\widehat{j}(X,Y) = D^G(Y)^{1/2}\sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})} \Gamma_{\mathcal{O}}(X)\widehat{j}(\mathcal{O},Y).$$

  (Shalika when F is p-adic, Beuzart-Plessis when F Archimedean)

Geometric expansion: Lie algebra variant

- Taking the regular germ, for any $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$,

$$c_{\theta_f,\mathcal{O}}(0) = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta_{\widehat{f}}(X)\Gamma_{\mathcal{O}}(X)dX.$$

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- Need explicit formula for $\Gamma_{\mathcal{O}}(X)$.

Regular germ formula

Theorem (L.)

For G a quasi-split reductive algebraic group, $X \in \mathfrak{g}^{\mathrm{rss}}(F)$ and $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, set $T_G = G_X$. Then

$$\Gamma_{\mathcal{O}}(X) = \begin{cases} 1, & \mathrm{inv}(X)\mathrm{inv}(T_G) = \mathrm{inv}_{T_G}(\mathcal{O}), \\ 0, & \text{otherwise.} \end{cases}$$

When F is p-adic the result was already proved by D. Shelstad.

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- $\mathrm{inv}(T_G)$ is connected with the Langlands-Shelstad transfer factor Δ_{I}.
- $\mathrm{inv}(X)$ is connected with the Langlands-Shelstad transfer factor Δ_{II}.

Relation with the Kostant's sections

Based on a result of Kottwitz, we also prove the following theorem.

Theorem (L.)

$\Gamma_{\mathcal{O}}(X) = 1$ *if and only if the* $G(F)$*-orbit of* X *and* $\mathcal{O}$ *lie in the* $G(F)$*-orbit of a common Kostant's section.*

- Kostant constructed a section for $\mathfrak{g} \to \mathfrak{g} /\!/ G \simeq \mathfrak{t}/W$, whose image in $\mathfrak{g}$ contains only regular elements, and meets every regular stable $\mathrm{Ad}(G)$-orbit exactly once.

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- $\mathfrak{g}^{\mathrm{reg}} := \{X \in \mathfrak{g} | \quad \dim \mathrm{Cent}_{\mathfrak{g}}(X) = \dim \mathfrak{t}\}$. Regular elements are not necessarily semi-simple, e.g. regular nilpotent elements.
- The restriction of $\mathfrak{g} \to \mathfrak{t}/W$ to an $\mathrm{Ad}(G)$-orbit of a Kostant's section is a smooth submersion. The measures on the fibers are given by the relevant orbital integrals.

Geometric expansion: Lie algebra variant

- $\{\theta_f | \quad f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))\}$ is dense in the space of quasi-characters on $\mathfrak{g}(F)$ when $F = \mathbb{R}$ or p-adic. Moreover, $\widehat{\theta}_f = \theta_{\widehat{f}}$ (Beuzart-Plessis).

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- So reduce to show for any quasi-character θ,

$$J^{\mathrm{Lie}}(\theta) = J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta)$$

with

$$J^{\mathrm{Lie}}(\theta) = \int_{\Gamma(\Xi+\mathfrak{h}^{\perp})} D^G(X)^{1/2}\widehat{\theta}(X)dX,$$

$$J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta) = \int_{\Gamma^{\mathrm{Lie}}(G,H)} c_\theta(X)D^G(X)^{1/2}\Delta(X)^{-1/2}dX.$$

Geometric expansion: Lie algebra variant

- By homogeneity of $J^{\mathrm{Lie}}(\theta)$ and $J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta)$, i.e.

$$J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta_\lambda) = |\lambda|^{\delta(G)/2} J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta)$$

  where $\theta_\lambda(X) = \theta(\lambda^{-1}X)$,
  we can show:

$$J^{\mathrm{Lie}}(\theta) - J^{\mathrm{Lie}}_{\mathrm{geom}}(\theta) = \sum_{\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})} c_{\mathcal{O}} c_{\theta,\mathcal{O}}(0)$$

  for some constant $c_{\mathcal{O}} \in \mathbb{C}$.

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- $$c_{\theta,\mathcal{O}}(0) = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\widehat{\theta}(X)\Gamma_{\mathcal{O}}(X)dX.$$

Geometric expansion: Lie algebra variant

- To prove that $c_{\mathcal{O}} = 0$, for $X \in \mathfrak{g}^{\mathrm{rss}}(F)$ with

$$\mathfrak{t}_X^G \cap \Gamma(G, H) = \{0\},$$

attach it with quasi-character $\theta_X = \widehat{j}(X, \cdot)$ supported on $\mathfrak{t}_X^G$,

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- In particular, $c_{\theta_X, \mathcal{O}}(0) = \Gamma_{\mathcal{O}}(X)$, and

$$J_{\mathrm{geom}}^{\mathrm{Lie}}(\theta_X) = \begin{cases} \Gamma_{\mathcal{O}}(X), & |\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})| = 1, \\ \Gamma_{\mathcal{O}_{\nu_0}}(X), & \dim V \text{ even} \geq 4, \\ \Gamma_{\mathcal{O}_{-\nu_0}}(X), & \dim W \text{ even} \geq 4. \end{cases}$$

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- Similarly,
$$J^{\mathrm{Lie}}(\theta_X) = \begin{cases} c_{\theta_X, \mathcal{O}}(0) = \Gamma_{\mathcal{O}}(X), & X \in \Xi + \mathfrak{h}^{\perp}, \Gamma_{\mathcal{O}}(X) = 1, \\ 0, & \text{Otherwise}, \end{cases}$$

Geometric expansion: Lie algebra variant

- When $|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})| = 1$, for $X \in \mathfrak{t}_{\mathrm{qd}}^{\mathrm{rss}}$, $\Gamma_{\mathcal{O}}(X) = 1$ identically, and $X \in \Xi + \mathfrak{h}^{\perp}$, therefore

$$J_{\mathrm{geom}}^{\mathrm{Lie}}(\theta_X) = J^{\mathrm{Lie}}(\theta_X) = 1.$$

Hence $c_{\mathcal{O}} = 0$.

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- Therefore $c_{\mathcal{O}} = 0$ for any $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$.

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- To prove the above claim, we need explicit formula for $\Gamma_{\mathcal{O}}(X)$, and find relation between the formula and $X \in \Xi + \mathfrak{h}^{\perp}$.

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- We compute the invariants $\frac{\mathrm{inv}(T_G)\mathrm{inv}(X)}{\mathrm{inv}_{T_G}(\mathcal{O})}$ explicitly for any $X \in \mathfrak{g}^{\mathrm{rss}}$ without eigenvalue 0, following the work of Waldspurger.

Thank you!