A Chabauty-Coleman bound for surfaces in abelian threefolds

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Chabauty's theorem

Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$ and Jacobian of J.

- Mordell's conjecture '22: $C(\mathbb{Q})$ is finite.
- Chabauty's theorem '41: If $\mathrm{rk}\, J(\mathbb{Q}) < g$, then $C(\mathbb{Q})$ is finite.
- Faltings's theorem '83: Mordell's conjecture is true.

Nevertheless, Chabauty's rank condition

$$\mathrm{rk}\, J(\mathbb{Q}) < g$$

holds quite often in practice, and the proof of Chabauty's theorem is much simpler.

Logarithms, exponentials

Let A be an abelian variety over $\mathbb{Q}_p$. This is a p-adic Lie group and there is a classical theory of logarithm and exponential map well-documented in Bourbaki's *Lie grous and Lie algebras*, Ch. III.

- One gets an analytic group morphism

$$\mathrm{Log} : A(\mathbb{Q}_p) \to \mathbb{Q}_p^g$$

  which is an isomorphism of Lie groups near e.
- Locally, it has an inverse

$$\mathrm{Exp} : B_0(r) \subseteq \mathbb{Q}_p^g \to U \subseteq A(\mathbb{Q}_p), \quad \text{for a suitable } r > 0.$$

Sketch of Chabauty's proof

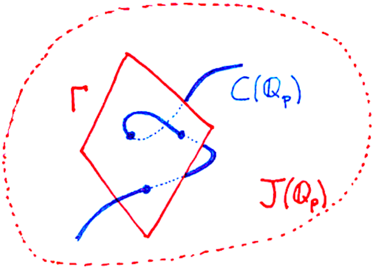
Take $x_0 \in C(\mathbb{Q})$, if any. Embed C into J via $x \mapsto [x - x_0]$. Let $r = \mathrm{rk}\, J(\mathbb{Q})$.

- Let Γ be the p-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$. It is a p-adic Lie subgroup of $J(\mathbb{Q}_p)$.
- The theory of Exp and Log on p-adic Lie groups implies

$$\dim \Gamma \leq r$$

- Hence, $\dim \Gamma < g = \dim J$.
- $C(\mathbb{Q}_p)$ generates $J(\mathbb{Q}_p)$, so it is not contained in Γ.
- It follows that $C(\mathbb{Q}_p) \cap \Gamma$ is finite.
- Finally, note that $C(\mathbb{Q}) = C(\mathbb{Q}_p) \cap J(\mathbb{Q}) \subseteq C(\mathbb{Q}_p) \cap \Gamma$. □

Coleman's bound



Coleman:

Reinterpret $\Gamma \cap C(\mathbb{Q}_p)$ as zeros of p-adic analytic functions on $C(\mathbb{Q}_p)$ constructed by integrating differentials.

Theorem (Coleman '85)

Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$ and Jacobian J. Let $p > 2g$ be a prime of good reduction. If $\mathrm{rk}\, J(\mathbb{Q}) \leq g - 1$, then

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + 2g - 2.$$

Developments around Chabauty-Coleman

- Explicit computations.
- Progress towards uniformity [Stoll], [Katz, Rabinoff, Zureick-Brown]
- Non-abelian extensions after M. Kim.
- Specially, a version of the quadratic case is now practical [Balakrishnan, Besser, Müller], [Balakrishnan, Dogra]
- Spectacular applications: $X_s(13)$ [Balakrishnan, Dogra, Müller, Tuitman, Vonk]

An elusive problem: Chabauty-Coleman beyond curves

What about a Chabauty-Coleman bound for X a higher dimensional subvariety of an abelian variety A ? Say, over $\mathbb{Q}$ and assuming

$$\mathrm{rk}\, A(\mathbb{Q}) + \dim X \leq \dim A.$$

So far, only explored when $A = J$ the Jacobian of a curve $C \subseteq J$ and

$$X = C + C + ... + C, \quad d \text{ times. (Essentially } \mathrm{Sym}^d C)$$

- [Klassen '93] Finiteness on a p-adic open set.
- [Siksek '09] Over number fields. Practical procedure for computations.
- [Park '16], [Vemulapalli, Wang '17]: A conditional bound (not of Coleman type) assuming the existence of suitable differentials.

Chabauty-Coleman beyond curves

Heuristic

- Let A be an abelian variety and $X \subseteq A$ a sub-variety, both over $\mathbb{Q}$.
- Let $\Gamma = \overline{A(\mathbb{Q})} \subseteq A(\mathbb{Q}_p)$. This is a p-adic Lie subgroup.
- If X generates A and

  $$\mathrm{rk}\, A(\mathbb{Q}) + \dim X \leq \dim A \quad \text{(Chabauty rank condition)}$$

  then we might expect that $X(\mathbb{Q}_p) \cap \Gamma$ is finite.
  (Not really... e.g. X might contain an elliptic curve of positive rank).
- Finally, note that $X(\mathbb{Q}) = X(\mathbb{Q}_p) \cap A(\mathbb{Q}) \subseteq X(\mathbb{Q}_p) \cap \Gamma$.
- Then we would love to express $X(\mathbb{Q}_p) \cap \Gamma$ as zeros of p-adic analytic functions on $X(\mathbb{Q}_p)$ to generalize Coleman's bound!

When should we expect finiteness of $X(\mathbb{Q})$?

A smooth projective complex variety M is (Brody) *hyperbolic* if every holomorphic map $f : \mathbb{C} \to M$ is constant.

- If $M = C$ is a curve, this exactly means $g(C) \geq 2$ (Picard).
- More generally, if M is contained in an abelian variety A, this means that M does not contain translates of positive dimensional abelian subvarieties of A (Green, Kawamata)
- Assume M is defined over $\mathbb{Q}$. General conjectures of Bombieri, Lang, and Vojta predict that if M is hyperbolic, then $M(\mathbb{Q})$ is finite. So the problem of bounding $\#M(\mathbb{Q})$ makes sense.
- For curves: C hyperbolic implies $C(\mathbb{Q})$ finite (Faltings)
- For subvarieties of abelian varieties: X hyperbolic implies $X(\mathbb{Q})$ finite (Faltings '91, generalizing methods introduced by Vojta).

Main result over $\mathbb{Q}$ (there is a version over number fields)

Theorem (Caro - P.)

- *Let A be an abelian variety of dimension 3 with $\mathrm{rk}\, A(\mathbb{Q}) = 1$.*
- *Let $X/\mathbb{Q}$ be a smooth projective hyperbolic surface contained in A.*
- *Let $p > 15 \cdot c_1^2(X)^2$ be a prime of good reduction such that $X \otimes \mathbb{F}_p^{alg}$ does not contain elliptic curves ("hyperbolic reduction").*

Then $\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + (p + 4\sqrt{p} + 8) \cdot c_1^2(X)$.

Remark. $c_1^2(X) = (K_X.K_X)$. This is the first Chern number of X.

Examples

Corollary

Let $A/\mathbb{Q}$ be an abelian threefold with $\mathrm{rk}\, A(\mathbb{Q}) = 1$ and $\mathrm{End}(A_{\mathbb{C}}) = \mathbb{Z}$. Let

$$\mathcal{P} = \{p : A_{\mathbb{F}_p} \text{ is good and absolutely simple}\}$$

Then for every smooth surface $X \subseteq A$ over $\mathbb{Q}$ and every $p \in \mathcal{P}$ of good reduction for X with $p > 15c_1^2(X)^2$, we have

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + (p + 4\sqrt{p} + 8) \cdot c_1^2(X).$$

- $\mathcal{P}$ has density 1 in the primes [Chavdarov '97].
- The conditions on hyperbolicity are automatically satisfied.
- Abundant examples; e.g. $A =$ the Jacobian of $y^2 = x^7 - x - 1$.

The shape of the bound

Coleman's bound for hyperbolic ($g \geq 2$) curves:

$$\#C(\mathbb{Q}) \leq \underbrace{\#C(\mathbb{F}_p)}_{\approx p} + \underbrace{2g-2}_{=c_1(C)}$$

Our bound for hyperbolic surfaces in abelian threefolds:

$$\#X(\mathbb{Q}) \leq \underbrace{\#X(\mathbb{F}_p)}_{\approx p^2} + \underbrace{(p + 4\sqrt{p} + 8) \cdot c_1^2(X)}_{\approx p \cdot c_1^2(X)}.$$

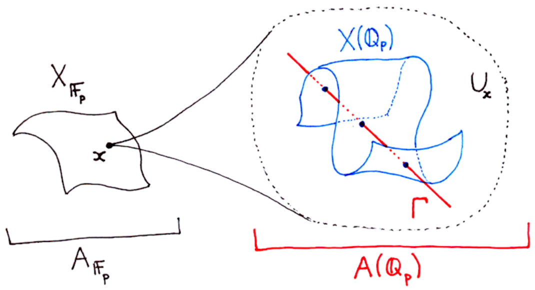
- In both cases the main term is counting points mod p, and the error term is a lower order contribution coming from the canonical class.
- It is tempting to conjecture that this is the general pattern!

Sketch of proof: setup

- $\Gamma = \overline{A(\mathbb{Q})}$ is a p-adic analytic 1-parameter subgroup of $A(\mathbb{Q}_p)$. Note:

$$X(\mathbb{Q}) = X(\mathbb{Q}_p) \cap A(\mathbb{Q}) \subseteq X(\mathbb{Q}_p) \cap \Gamma.$$

- Reduction map: $\mathrm{red} : A(\mathbb{Q}_p) \to A(\mathbb{F}_p)$. For each residue disk $U_x = \mathrm{red}^{-1}(x)$ with $x \in X(\mathbb{F}_p)$ we want to bound $\# X(\mathbb{Q}_p) \cap \Gamma \cap U_x$.



Sketch of proof: how to bound $\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x$?

- Parametrize the analytic 1-parameter subgroup $\gamma : p\mathbb{Z}_p \to \Gamma \cap U_x$.
- Let f be a local equation for X on U_x. Then $f \circ \gamma(z)$ is a p-adic power series and

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq n_0(f \circ \gamma(z), 1/p).$$

- $f \circ \gamma(z) = \sum_n a_n z^n \in \mathbb{Q}_p[[z]]$ with controlled growth of $|a_n|$.
- ***p*-adic analysis:** To bound $n_0(f \circ \gamma(z), 1/p)$ we "just" need a small index N with $|a_N|$ large, say $|a_N| \geq 1$.

This last requirement is **very difficult**.
The existing methods don't seem to help. We need some additional theory.

ω-integrality: an algebraic version of ODE's

Let k be a field, S and V are k-schemes and $\omega \in H^0(S, \Omega^1_{S/k})$. A k-morphism $\phi : V \to S$ is ω-**integral** if the composition

$$\phi^\bullet : H^0(S, \Omega^1_{S/k}) \to H^0(S, \phi_*\phi^*\Omega^1_{S/k}) = H^0(V, \phi^*\Omega^1_{S/k}) \to H^0(V, \Omega^1_{V/k})$$

satisfies $\phi^\bullet(\omega) = 0$.

- For varieties over $\mathbb{C}$, this is very useful for proving hyperbolicity and to study curves in varieties.
- We'll use this on non-reduced schemes in positive characteristic, so the analytic intuition over $\mathbb{C}$ is not very helpful.

ω-integrality

The notion of ω-integrality is implicit in classical works by Nakai (and then forgotten for a while). Very useful in the context of hyperbolicity:

- Bogomolov: Finiteness of curves of geometric genus 0 and 1 on certain general type surfaces. Then by McQuillan and others.
- Vojta: Explicit version of Bogomolov for the surfaces in Büchi's problem to fully compute the curves of geometric genus 0 and 1. (This problem is motivated by logic!)
- Garcia-Fritz: Purely algebraic extension of Vojta's explicit approach. We use her methods adapted to positive characteristic.

Remark. Going from finiteness to **explicit** finiteness is not obvious. Think about rational points on curves!

Large coefficient in low degree: the overdetermined method

- Take $\omega_1, \omega_2 \in H^0(A, \Omega^1_{A/\mathbb{Q}_p})$ independent, with nice reduction mod p, such that the p-adic 1-parameter subgroup

$$\gamma : p\mathbb{Z}_p \to A(\mathbb{Q}_p)$$

  is ω_i-integral for both $i = 1, 2$.

- Express $\gamma(z)$ as power series and reduce mod z^{m+1} with $\boxed{m < p}$.
- We get a closed immersion over $\mathbb{Q}_p$

$$\phi_m^0 : \mathrm{Spec}\, \mathbb{Q}_p[z]/(z^{m+1}) \to A_{\mathbb{Q}_p}$$

  which is ω_i-integral for $i = 1, 2$, and everything reduces nicely mod p.

- This gives a similar map $\phi_m : \mathrm{Spec}\, k[z]/(z^{m+1}) \to A_k$ over $k = \mathbb{F}_p^{alg}$.

Large coefficient in low degree: the overdetermined method

- Recall: f local equation for X on U_x, with $x \in X(\mathbb{F}_p)$.
- **Key observation**. If $f \circ \gamma$ has p-adically small coefficients up to degree m, we would get $f \circ \phi_m^0 \mod p = 0$, implying that ϕ_m actually is a closed immersion into X_k, not just A_k.
- Let $w_1, w_2 \in H^0(X_k, \Omega^1_{X_k/k})$ be obtained by reducing ω_i mod p and restricting to X_k. We need an upper bound for any m that satisfies:

  "*There is a closed immersion* $\phi : \mathrm{Spec}\, k[z]/(z^{m+1}) \to X_k$ *supported at* x *which is* w_i*-integral for both* $i = 1, 2$."
- **This is an overdetermined ODE!** A large m should be rare.
- **The "overdetermined" bound**: We bound m in terms of the geometry of $D = \mathrm{div}(w_1 \wedge w_2)$.

A bound (it's ugly —sorry)

Let $k = \mathbb{F}_p^{alg}$ (or any alg. closed field) and $V_m = \mathrm{Spec}\, k[z]/(z^{m+1})$.

Lema (The overdetermined bound)

Let S be a smooth surface over k, let $x \in S$, let $w_1, w_2 \in H^0(S, \Omega^1_{S/k})$ be independent over $\mathcal{O}_S$ and let $D = div(w_1 \wedge w_2)$. Write $D = \sum_{j=1}^{\ell} a_j C_j$ with C_j irreducible curves and let $\nu_j : \tilde{C}_j \to S$ be the normalizations. Let $\phi : V_m \to S$ be a closed immersion supported at x. If ϕ is w_i-integral for both $i = 1, 2$ then

$$m \leq \sum_{j=1}^{\ell} \sum_{y \in \nu_j^{-1}(x)} a_j \cdot \left(\mathrm{ord}_y(\nu_j^\bullet(w_i)) + 1\right).$$

Remark. Some ν_j might be w_i-integral. That's fine: $\mathrm{ord}_y(0) = +\infty$. However, this case is useless and we must avoid it (technical point).

An example (assume $\mathrm{char}(k) \neq 2, 3$)

- In $S = \mathbb{A}^2 = \mathrm{Spec}\, k[s,t]$ take $x = (0,0)$ and the differentials

$$w_1 = ds + t^2 dt, \quad w_2 = ds + s^2 dt$$

- Then $w_1 \wedge w_2 = (s^2 - t^2) ds \wedge dt = (s-t)(s+t) ds \wedge dt$ and we get $D = C_1 + C_2$ with $C_1 = \{s = t\}$, $C_2 = \{s = -t\}$.
- We have the closed immersion $\phi : V_2 \to S$ supported at x:

$$k[s,t] \to k[z]/(z^3), \quad s \mapsto 0, \quad t \mapsto z.$$

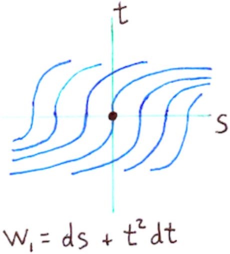
- $\phi : V_2 \to S$ is w_i-integral ($i = 1, 2$). That is, w_1, w_2 have image 0 in

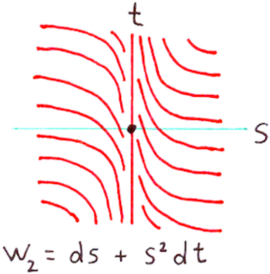
$$\Omega_{(k[z]/(z^3))/k} = (k[z]/(z^3, 3z^2))dz = (k[z]/(z^2))dz.$$

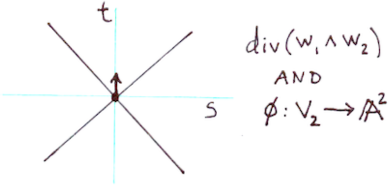
- For w_1 (and similarly for w_2) the bound is sharp:

$$(\mathrm{ord}_x(w_1|_{C_1}) + 1) + (\mathrm{ord}_x(w_1|_{C_2}) + 1) = (0+1) + (0+1) = 2.$$

An example (assume $\mathrm{char}(k) \neq 2, 3$)







Sketch of proof: applying the overdetermined method

- p-adic analysis and the "overdetermined method" give a bound

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq n_0(f \circ \gamma(z), 1/p) \leq \frac{p-1}{p-2}(m(x)+1)$$

where $m(x)$ is the largest m with an overdetermined (for w_1, w_2) closed immersion $\phi_m : \mathrm{Spec}\, k[z]/(z^{m+1}) \to X_k$ supported at x.

- Our theory of overdetermined ω-integrality in characteristic p gives a bound for $m(x)$ in terms of $D = div(w_1 \wedge w_2)$ on $X_{\mathbb{F}_p}$.

Remark. Proving "$w_1 \wedge w_2 \neq 0$" in characteristic p is difficult.

Sketch of proof: two very different cases

- $\boxed{x \notin supp(D)}$ Our bound gives $m(x) \leq \sum_{\emptyset}(...) = 0$. Hence

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq \frac{p-1}{p-2}(m(x)+1) = \frac{p-1}{p-2} < 2.$$

  So we get $\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq 1$

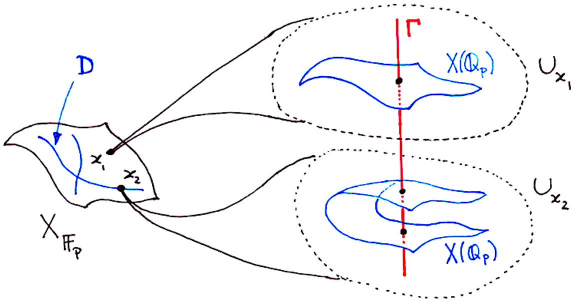
- $\boxed{x \in supp(D)}$ The "overdetermined bound" is much more complicated to apply. It needs the Riemann hypothesis for singular curves, intersection theory computations, controlling singularities of D, "weak Lefschetz" properties in positive characteristic, etc.

Remark. $D = div(w_1 \wedge w_2)$ is a canonical divisor on $X_{\mathbb{F}_p}$, hence $c_1^2(X) = (D.D)$ shows up in the bounds of the second case.

Sketch of proof: putting things together

At the end, adding the contribution of each U_x for $x \in X(\mathbb{F}_p)$ gives

$$\#\left(X(\mathbb{Q}_p) \cap \Gamma\right) < \underbrace{\#X(\mathbb{F}_p)}_{x \in X(\mathbb{F}_p) - D} + \underbrace{(p + 4\sqrt{p} + 8) \cdot c_1^2(X)}_{x \in D(\mathbb{F}_p)}.$$



Thanks for your attention.