Cohomology sheaves of stacks of shtukas

Cong Xue

CNRS and IMJ-PRG

19 March 2021

Introduction

In this talk, we will

- recall the definition of the stacks of shtukas and their cohomology sheaves
- talk about the finiteness and smoothness properties of the cohomology sheaves

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, $\operatorname{char} \mathbb{F}_q = p$. Let F be its function field.
Let G be a connected reductive group over F.

In the talk : to simplify, we only consider the case without level structure (i.e. everywhere unramified) and we suppose that G is split.

Let $\widehat{G}$ be the Langlands dual group of G over $\mathbb{Q}_\ell$, where $\ell \neq p$.

Let $\mathbb{A}$ be the ring of adeles of F and $\mathbb{O}$ be the ring of integral adeles. Let Z_G be the center of G. We fix a discrete subgroup Ξ in $Z_G(\mathbb{A})$ such that $Z_G(F)\backslash Z_G(\mathbb{A})/Z_G(\mathbb{O})\Xi$ is finite. When G is semisimple, we can take $\Xi = 1$.

We have the space of automorphic forms for the function field F :

$$C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi, \mathbb{Q}_\ell) = C_c(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$$

where Bun_G is the classifying stack of G-bundles over X.

Example : $G = GL_1$, the space of automorphic forms (here $\Xi \simeq \mathbb{Z}$) $C_c(\mathrm{Pic}_X(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$ has finite dimension.

Example : $G = SL_2$, $X = \mathbb{P}^1$, the space of automorphic forms $C_c(\mathrm{Bun}_{SL_2}(\mathbb{F}_q), \mathbb{Q}_\ell)$ has infinite dimension (because there are infinitely many rank 2 vector bundles of trivial determinant on X, such as $\mathcal{O}(n) \oplus \mathcal{O}(-n)$).

Stacks of shtukas : example for $G = GL_1$

$I = \{1, 2, \cdots, k\}$, $W = (w_i)_{i \in I}$ with $w_i \in \mathbb{Z}$.

The stack of shtukas associated to I and W is the fiber product (non empty iff $\sum_{i \in I} w_i = 0$) :

$$\begin{array}{ccc} \mathrm{Cht}_{GL_1, I, W} & \longrightarrow & \mathrm{Pic}_X \\ \downarrow \mathfrak{p} & & \downarrow \text{Lang's isogeny} \\ X^I & \xrightarrow{\text{Abel-Jacobi}} & \mathrm{Pic}_X \end{array} \qquad \begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathcal{L}^{-1} \otimes {}^{\tau}\mathcal{L} \end{array}$$

$$(x_i)_{i \in I} \quad \mapsto \quad \mathcal{O}_{X \times S}\left(\sum w_i x_i\right)$$

For any S affine scheme over $\mathbb{F}_q$, $\mathrm{Pic}_X(S) = \{\mathcal{L} \text{ line bundle on } X \times_{\mathbb{F}_q} S\}$, ${}^{\tau}\mathcal{L} := (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{L}$, where Frob_S is the absolute Frobenius over $\mathbb{F}_q$.

$$\mathrm{Cht}_{GL_1, I, W}(S) = \{(x_i)_{i \in I} \in X^I(S), \mathcal{L} \xrightarrow{\sim} {}^{\tau}\mathcal{L}(\textstyle\sum w_i x_i)\}.$$

A shtuka is a S-point of the stack of shtukas. The points x_i are called the paws of the shtuka. The morphism $\mathfrak{p}$ is called the morphism of paws.

Example : $I = \{1, 2\}$, $w_1 = 1$, $w_2 = -1$.

$$\begin{aligned} \mathrm{Cht}_{GL_1,I,W}(S) &= \{x_1, x_2 \in X(S), \mathcal{L} \xrightarrow{\sim} {}^{\tau}\mathcal{L}(x_1 - x_2)\} \\ &= \{x_1, x_2 \in X(S), \mathcal{L} \hookrightarrow \mathcal{L}(x_1) \hookleftarrow {}^{\tau}\mathcal{L}(x_1 - x_2)\} \\ &= \{x_1, x_2 \in X(S), \mathcal{L} \hookleftarrow \mathcal{L}(-x_2) \hookrightarrow {}^{\tau}\mathcal{L}(x_1 - x_2)\} \end{aligned}$$

When I is the empty set, $X^I = \operatorname{Spec} \mathbb{F}_q$, we have $\mathrm{Cht}_{GL_1,\emptyset} = \mathrm{Pic}_X(\mathbb{F}_q)$.

Example of Drinfeld's stacks of shtukas

$G = GL_n$, $I = \{1,2\}$, $W = \mathrm{St} \boxtimes \mathrm{St}^*$ with St the standard representation of $\widehat{G} = GL_n$ and St^* its dual. In the following we note ${}^\tau\mathcal{G} := (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{G}$.

$$\mathrm{Cht}^{(1,2)}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*}(S) := \{x_1, x_2 \in X(S), \mathcal{G}_0, \mathcal{G}_1 : \text{rk } n \text{ vector bundles on}$$

$$\text{on } X \times_{\mathbb{F}_q} S,\ \mathcal{G}_0 \overset{\phi_1}{\hookrightarrow} \mathcal{G}_1 \overset{\phi_2}{\hookleftarrow} {}^\tau\mathcal{G}_0 \text{ s.t. } \phi_1 \text{ isom outside } x_1, \phi_2 \text{ isom outside } x_2,$$

$$\mathcal{G}_1/\mathcal{G}_0 \text{ is an invertible sheaf on the graph of } x_1,$$

$$\mathcal{G}_1/{}^\tau\mathcal{G}_0 \text{ is an invertible sheaf on the graph of } x_2\}.$$

$$\mathrm{Cht}^{(2,1)}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*}(S) := \{x_1, x_2 \in X(S), \mathcal{G}'_0, \mathcal{G}'_1 : \text{rk } n \text{ vector bundles}$$

$$\text{on } X \times_{\mathbb{F}_q} S,\ \mathcal{G}'_0 \overset{\phi'_1}{\hookleftarrow} \mathcal{G}'_1 \overset{\phi'_2}{\hookrightarrow} {}^\tau\mathcal{G}'_0 \text{ s.t. } \phi'_1 \text{ isom outside } x_2, \phi'_2 \text{ isom outside } x_1,$$

$$\mathcal{G}'_0/\mathcal{G}'_1 \text{ is an invertible sheaf on the graph of } x_2,$$

$${}^\tau\mathcal{G}'_0/\mathcal{G}'_1 \text{ is an invertible sheaf on the graph of } x_1\}.$$

$$\mathrm{Cht}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*}(S) := \{x_1, x_2 \in X(S), \mathcal{G}_0 : \text{rk } n \text{ vector bundle on}$$

$$\text{on } X \times_{\mathbb{F}_q} S,\ \mathcal{G}_0 \overset{\phi}{\dashrightarrow} {}^\tau\mathcal{G}_0 \ \text{ s.t. } \phi \text{ isom outside } x_1 \text{ and } x_2,$$

$$\text{there exists a diagram } \mathcal{G}_0 \dashrightarrow \mathcal{G}_1 \dashrightarrow {}^\tau\mathcal{G}_0 \text{ as above } \}$$

We have the forgeting morphism

$$\mathrm{Cht}^{(1,2)}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*} \xrightarrow{\pi} \mathrm{Cht}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*} \to X^2$$

$$((x_1, x_2), \mathcal{G}_0 \hookrightarrow \mathcal{G}_1 \hookleftarrow {}^\tau\mathcal{G}_0) \mapsto ((x_1, x_2), \mathcal{G}_0 \dashrightarrow {}^\tau\mathcal{G}_0) \mapsto (x_1, x_2)$$

Outside the diagonal of X^2, π is an isomorphism. Fact : the morphism π is small.

Similarly for $\mathrm{Cht}^{(2,1)}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*} \to \mathrm{Cht}_{GL_n,\{1,2\},\mathrm{St}\boxtimes\mathrm{St}^*}$.

Stacks of shtukas : in general

Let $I = \{1, 2, \cdots, k\}$ be a finite set. Let W be a finite dim $\mathbb{Q}_\ell$-linear representation of $\widehat{G}^I$. Suppose $W = \boxtimes_{i \in I} W_i$, with W_i irreducible representation of $\widehat{G}$ of highest weight λ_i.

The stack of shtukas (defined by Drinfeld and Varshavsky) associated to I, W and order $(1, 2, \cdots, k)$ is the following fiber product

$$\begin{array}{ccc} \mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} & \longrightarrow & \mathrm{Bun}_G \\ \downarrow & & \downarrow {\scriptstyle (\mathrm{Id},\mathrm{Frob})} \\ \mathrm{Hecke}_{G,I,W}^{(1,2,\cdots,k)} & \longrightarrow & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array}$$

$$((x_i), \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} \mathcal{G}_k) \quad \mapsto \quad (\mathcal{G}_0, \mathcal{G}_k)$$

where $\mathrm{Hecke}_{G,I,W}^{(1,2,\cdots,k)}$ is the Hecke stack associated to I and W : ϕ_i is an isomorphism outside x_i, the relative position of $\mathcal{G}_{i-1}$ and $\mathcal{G}_i$ at the formal neighborhood of x_i is bounded by λ_i.

$$\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)}(S) = \{(x_i)_{i\in I} \in X^I(S),\ \mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{k-1} : G\text{-bundles}$$

$$\text{on } X \times_{\mathbb{F}_q} S,\ \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \dashrightarrow \mathcal{G}_{k-1} \xrightarrow{\phi_k} {}^{\tau}\mathcal{G}_0 \text{ s.t. } \phi_i \text{ as above.}\}$$

We have the forgeting morphism which is a small morphism

$$\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} \xrightarrow{\pi} \mathrm{Cht}_{G,I,W}$$

$$((x_i), \mathcal{G}_0 \dashrightarrow \mathcal{G}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{G}_{k-1} \dashrightarrow {}^{\tau}\mathcal{G}_0) \mapsto ((x_i), \mathcal{G}_0 \dashrightarrow {}^{\tau}\mathcal{G}_0)$$

In the following, to simplify, we will omit the upper index because the results are true for any upper index.

$\mathrm{Cht}_{G,I,W}$ is a Deligne-Mumford algebraic stack locally of finite type.

We define

$$\mathrm{Cht}_{G,I,W\oplus W'} := \mathrm{Cht}_{G,I,W} \bigcup \mathrm{Cht}_{G,I,W'}$$

We can define $\mathrm{Cht}_{G,I}$ which is an inductive limit of algebraic stacks.

Satake perverse sheaf over stack of shtukas

We have the morphism of paws

$$\mathfrak{p} : \mathrm{Cht}_{G,I,W} \to X^I$$

In general, the stack of shtukas $\mathrm{Cht}_{G,I,W}$ is not smooth. We have a canonical perverse sheaf $\mathrm{Sat}_{G,I,W}$ over $\mathrm{Cht}_{G,I,W}$, which comes from the geometric Satake equivalence (Mirkovic-Vilonen).

When W is irreducible, $\mathrm{Sat}_{G,I,W}$ is isomorphic to the intersection complex (with coefficient in $\mathbb{Q}_\ell$ and the perverse normalization relative to X^I).
Example : when $\mathrm{Cht}_{G,I,W}$ is smooth and W irreducible,
$\mathrm{Sat}_{G,I,W} = \text{IC-sheaf} = \mathbb{Q}_\ell[d]$, where $d = \dim \mathrm{Cht}_{G,I,W} - \dim X^I$.

$$\mathrm{Sat}_{G,I,W \oplus W'} := \mathrm{Sat}_{G,I,W} \bigoplus \mathrm{Sat}_{G,I,W'}$$

Remark : we can directly define $\mathrm{Sat}_{G,I,W}$ over $\mathrm{Cht}_{G,I}$. The stack $\mathrm{Cht}_{G,I,W}$ is the support of $\mathrm{Sat}_{G,I,W}$.

Harder-Narasimhan stratification

To simply the notation, suppose that G is semisimple. The stack of shtukas $\mathrm{Cht}_{G,I,W}$ is locally of finite type but not necessarily of finite type.
Example : recall that $\mathrm{Bun}_{SL_2}(\mathbb{F}_q)$ is infinite.

One way to define the Harder-Narasimhan stratification : for any μ dominant coweight of G, we have an open substack in Bun_G :

$$\mathrm{Bun}_G^{\leq \mu} = \{G\text{-bundle } \mathcal{G}_0, \text{ "the Harder-Narasimhan filtration" of } \mathcal{G}_0 \leq \mu\}$$

We define the truncated stack of shtukas as the fiber product :

$$\begin{array}{ccc} \mathrm{Cht}_{G,I,W}^{\leq \mu} & \xrightarrow{\text{open}} & \mathrm{Cht}_{G,I,W} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G^{\leq \mu} & \xrightarrow{\text{open}} & \mathrm{Bun}_G \end{array} \qquad \begin{array}{c} ((x_i), \mathcal{G}_0 \dashrightarrow \mathcal{G}_1 \cdots \dashrightarrow {}^{\tau}\mathcal{G}_0) \\ \downarrow \\ \mathcal{G}_0 \end{array}$$

The open substack $\mathrm{Cht}_{G,I,W}^{\leq \mu}$ is of finite type. And we have

$$\mathrm{Cht}_{G,I,W} = \bigcup_{\mu} \mathrm{Cht}_{G,I,W}^{\leq \mu}$$

Cohomology sheaves of the stack of shtukas

Recall that we have the morphism of paws $\mathfrak{p} : \mathrm{Cht}_{G,I,W} \to X^I$. We define the degree $j \in \mathbb{Z}$ truncated cohomology sheaf

$$\mathcal{H}^{j,\leq\mu}_{G,I,W} := R^j\mathfrak{p}_!(\mathrm{Sat}_{G,I,W}\,|_{\mathrm{Cht}^{\leq\mu}_{G,I,W}})$$

It is a constructible $\mathbb{Q}_\ell$-sheaf over X^I. Cohomology sheaves are concentrated in degree $j \in [-d, d]$ where $d = \dim \mathrm{Cht}_{G,I,W} - \dim X^I$.

For $\mu_1 \leq \mu_2$, we have an open immension

$$\mathrm{Cht}^{\leq\mu_1}_{G,I,W} \hookrightarrow \mathrm{Cht}^{\leq\mu_2}_{G,I,W}$$

It induces a morphism of sheaves

$$\mathcal{H}^{j,\leq\mu_1}_{G,I,W} \to \mathcal{H}^{j,\leq\mu_2}_{G,I,W}.$$

We define the degree j cohomology sheaf as the inductive limit

$$\mathcal{H}^{j}_{G,I,W} := \varinjlim_{\mu} \mathcal{H}^{j,\leq\mu}_{G,I,W}.$$

Let η_I be the generic point of X^I. Let $\overline{\eta_I}$ be a geometric point over η_I. We define the truncated cohomology group $H^{j,\leq\mu}_{G,I,W} := \mathcal{H}^{j,\leq\mu}_{G,I,W}\Big|_{\overline{\eta_I}}$ and the cohomology group $H^{j}_{G,I,W} := \mathcal{H}^{j}_{G,I,W}\Big|_{\overline{\eta_I}}$.

When $I = \emptyset$ (empty set), $W = \mathbf{1}$ (trivial representation), we have $\mathrm{Cht}_{G,\emptyset,\mathbf{1}} = \mathrm{Bun}_G(\mathbb{F}_q)$ and $H^0_{G,\emptyset,\mathbf{1}} = C_c(\mathrm{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell)$.

In general, $H^j_{G,I,W}$ is a $\mathbb{Q}_\ell$-vector space of possibly infinite dimension, equiped with

- an action of the Hecke algebra $\mathscr{H}_G := C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q}_\ell)$ by the Hecke correspondences, which doesn't preserve $H^{j,\leq\mu}_{G,I,W}$
- an action of $\pi_1(\eta_I, \overline{\eta_I})$ (evident), which preserves $H^{j,\leq\mu}_{G,I,W}$,
- an action of the partial Frobenius morphisms (one of the key properties of stack of shtukas), which doesn't preserve $H^{j,\leq\mu}_{G,I,W}$

Partial Frobenius morphisms : an example

Consider Drinfeld's stacks of shtukas. Let $G = GL_n$, $I = \{1, 2\}$, $W = St \boxtimes St^*$. Let Frob : $X \to X$ be the absolute Frobenius.

$$(\mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xleftarrow{\phi_2} {}^\tau\mathcal{G}_0) \mapsto (\mathcal{G}_1 \xleftarrow{\phi_2} {}^\tau\mathcal{G}_0 \xrightarrow{{}^\tau\phi_1} {}^\tau\mathcal{G}_1) \mapsto ({}^\tau\mathcal{G}_0 \xrightarrow{{}^\tau\phi_1} {}^\tau\mathcal{G}_1 \xleftarrow{{}^\tau\phi_2} {}^\tau{}^\tau\mathcal{G}_0)$$

$$\begin{array}{ccccc} \mathrm{Cht}_{G,I,W}^{(1,2)} & \xrightarrow{\mathrm{Frob}_{\{1\}}} & \mathrm{Cht}_{G,I,W}^{(2,1)} & \xrightarrow{\mathrm{Frob}_{\{2\}}} & \mathrm{Cht}_{G,I,W}^{(1,2)} \\ \downarrow \mathfrak{p} & & \downarrow \mathfrak{p} & & \downarrow \mathfrak{p} \\ X^2 & \xrightarrow{\mathrm{Frob}_{\{1\}}} & X^2 & \xrightarrow{\mathrm{Frob}_{\{2\}}} & X^2 \end{array}$$

$$(x_1, x_2) \mapsto (\mathrm{Frob}(x_1), x_2) \mapsto (\mathrm{Frob}(x_1), \mathrm{Frob}(x_2))$$

$$\mathrm{Frob}_{\{2\}} \circ \mathrm{Frob}_{\{1\}} = \text{ total Frobenius on } \mathrm{Cht}_{G,I,W}^{(1,2)}$$

Partial Frobenius morphisms : in general

In general, let $I = \{1, 2, \cdots, k\}$ and W an irreducible representation of $\widehat{G}^I$.

$(\mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \mathcal{G}_{k-1} \xrightarrow{\phi_k} {}^\tau\mathcal{G}_0) \mapsto (\mathcal{G}_1 \xrightarrow{\phi_2} \mathcal{G}_2 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_k} {}^\tau\mathcal{G}_0 \xrightarrow{{}^\tau\phi_1} {}^\tau\mathcal{G}_1)$

$$\begin{array}{ccc} \mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} & \xrightarrow{\mathrm{Frob}_{\{1\}}} & \mathrm{Cht}_{G,I,W}^{(2,\cdots,k,1)} \\ \downarrow \mathfrak{p} & & \downarrow \mathfrak{p} \\ X^I & \xrightarrow{\mathrm{Frob}_{\{1\}}} & X^I \end{array}$$

$$(x_1, x_2, \cdots, x_k) \mapsto (\mathrm{Frob}(x_1), x_2, \cdots, x_k)$$

The composition $\mathrm{Frob}_{\{1\}} \circ \cdots \circ \mathrm{Frob}_{\{k\}}$ is the total Frobenius on $\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)}$.

We have a canonical morphism :

$$\mathrm{Frob}_{\{1\}}^* \mathrm{Sat}_{G,I,W}^{(2,\cdots,k,1)} \xrightarrow{\sim} \mathrm{Sat}_{G,I,W}^{(1,2,\cdots,k)} \quad (\star)$$

Recall that the morphism $\mathrm{Cht}_{G,I,W}^{(1,2,\cdots,k)} \xrightarrow{\pi} \mathrm{Cht}_{G,I,W}$ is small. Fact : the cohomology sheaves of stacks of shtukas are independent of the upper index (the fact comes from a similar argument of small morphisms of Beilinson-Drinfeld affine grassmanians). Thus the cohomological correspondence for $(\star)$ induces a partial Frobenius morphism :

$$F_{\{1\}} : \mathrm{Frob}_{\{1\}}^* \mathcal{H}_{G,I,W}^{j,\leq\mu} \to \mathcal{H}_{G,I,W}^{j,\leq\mu+\kappa}$$

Similarly, we have $F_{\{2\}}, \cdots, F_{\{k\}}$.

The composition $F_{\{1\}} \circ \cdots \circ F_{\{k\}}$ is the total Frobenius morphism (composed with an augmentation of μ). The $F_{\{i\}}$ are called the partial Frobenius morphisms.

Taking the inductive limit, we have isomorphisms

$$F_{\{i\}} : \mathrm{Frob}_{\{i\}}^* \mathcal{H}_{G,I,W}^{j} \xrightarrow{\sim} \mathcal{H}_{G,I,W}^{j}$$

Drinfeld's lemma and the work of V. Lafforgue

Recall that F is the function field of X. Let $\eta = \operatorname{Spec} F$ be the generic point of X and $\overline{\eta} = \operatorname{Spec} \overline{F}$ be a geometric point over η. Note that $\pi_1(\eta, \overline{\eta}) = \mathrm{Gal}(\overline{F}/F)$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^{\mathrm{geo}}(\eta_I, \overline{\eta_I}) & \to & \pi_1(\eta_I, \overline{\eta_I}) & \to & \widehat{\mathbb{Z}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1^{\mathrm{geo}}(\eta, \overline{\eta})^I & \to & \pi_1(\eta, \overline{\eta})^I & \to & \widehat{\mathbb{Z}}^I & \to & 1 \end{array}$$

Drinfeld's lemma ($\mathbb{Z}_\ell$-version) (proved in [Drinfeld 89] and recalled in [V. Lafforgue]) : if a finite type $\mathbb{Z}_\ell$-module is equiped with an action of $\pi_1(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms, then it is equiped with an action of $\pi_1(\eta, \overline{\eta})^I$.

V. Lafforgue defined Hecke-finite cohomology $H^{j, \mathrm{Hf}}_{G,I,W} \subset H^j_{G,I,W}$ (a sub $\mathbb{Q}_\ell$-vector space). By the Eichler-Shimura relations, $H^{j, \mathrm{Hf}}_{G,I,W}$ is an inductive limit of finite type $\mathbb{Z}_\ell$-modules which are equiped with an action of the partial Frobenius morphisms. By Drinfeld's lemma, $H^{j, \mathrm{Hf}}_{G,I,W}$ is equipped with an action of $\mathrm{Gal}(\overline{F}/F)^I$.

Let $C_c^{\text{cusp}} \subset C_c(\text{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$ be the space of cuspidal automorphic forms. C_c^{cusp} is of finite dimension.

Excursion operator associated to I, W and $(\gamma_i)_{i\in I} \in \text{Gal}(\overline{F}/F)^I$:

$$C_c^{\text{cusp}} = H^{0,\text{Hf}}_{G,\emptyset,\mathbf{1}} \xrightarrow{\text{creation}} H^{0,\text{Hf}}_{G,I,W} \xrightarrow{(\gamma_i)_{i\in I}} H^{0,\text{Hf}}_{G,I,W} \xrightarrow{\text{annihilation}} H^{0,\text{Hf}}_{G,\emptyset,\mathbf{1}} = C_c^{\text{cusp}}$$

where "creation" and "annihilation" are constructed by using the functoriality of $H^0_{G,I,W}$ on W and the fusion (factorization).

Theorem (V. Lafforgue)

We have a canonical decomposition as $\mathscr{H}_G$-modules :
$C_c^{\text{cusp}} = \bigoplus_{\sigma:\text{Gal}(\overline{F}/F)\to\widehat{G}(\overline{\mathbb{Q}_\ell})} \mathfrak{H}_\sigma$, σ is $\widehat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy class of continuous, semisimple, everywhere unramified morphisms, the decomposition is compatible with the Satake isomorphism, i.e. for every place v of X, every irr rep V of $\widehat{G}$, the Hecke operator associated to v and V acts on $\mathfrak{H}_\sigma$ by multiplication by the scalar $\text{Tr}_V(\sigma(\text{Frob}_v))$.

More on Drinfeld's lemma

Drinfeld's lemma ($\mathbb{Q}_\ell$-version) (proved by Drinfeld, written in my paper [Finiteness]) : if a finite dim $\mathbb{Q}_\ell$-vector space is equiped with an action of Weil$(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms, then it is equiped with an action of Weil$(\eta, \overline{\eta})^I$.

An easy generalization is :
Drinfeld's lemma (Hecke-version) : if a finite type module over a local Hecke algebra (or over any finitely generated commutative $\mathbb{Q}_\ell$-algebra) is equiped with an action of Weil$(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms, then it is equiped with an action of Weil$(\eta, \overline{\eta})^I$.

Finiteness

My previous works : using the constant term morphisms for the cohomology groups of stacks of shtukas, we prove

Theorem 1

$H^j_{G,I,W}$ is a module of finite type over a local Hecke algebra.

Then by Drinfeld's lemma (Hecke-version), we have

Proposition 1

$H^j_{G,I,W}$ is equiped with an action of $\mathrm{Weil}(\eta,\overline{\eta})^I$.

Besides, using the constant term morphisms, we also prove

Theorem 2

(a) The $\mathbb{Q}_\ell$-v.s. $H^{j,\,\mathrm{Hf}}_{G,I,W}$ equals to $H^{j,\,\mathrm{cusp}}_{G,I,W}$ and they have finite dim.

(b) $H^{j,\,\mathrm{cusp}}_{G,I,W} = \bigoplus_{\sigma:\mathrm{Gal}(\overline{F}/F)\to\widehat{G}(\overline{\mathbb{Q}_\ell})}(H^{j,\,\mathrm{cusp}}_{G,I,W})_\sigma$, σ satisfying the conditions...

A new proof

In my recent work, which doesn't use the constant term morphisms at all, I give another proof of

Proposition 1

$H^j_{G,I,W}$ is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

and we prove

Proposition 2

The restriction $\mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^I}$ is constant over $(\overline{\eta})^I := \overline{\eta} \times_{\overline{\mathbb{F}_q}} \cdots \times_{\overline{\mathbb{F}_q}} \overline{\eta}$.

Remark : if $\mathcal{H}^j_{G,I,W}$ is of the form $\boxtimes_{i \in I} \mathcal{F}_i$, then both propositions are trivial.

Idea of the proof of Proposition 1 : we have $\mathcal{H}^{j}_{G,I,W}\Big|_{\overline{\eta_I}} := \varinjlim_{\mu} \mathfrak{M}_{\mu}$ with

$$\mathfrak{M}_{\mu} := \sum_{(n_i)_{i\in I}\in\mathbb{N}^I} (\otimes_{i\in I}\mathscr{H}_{G,v_i}) \cdot \big(\prod_{i\in I} \mathrm{Frob}_{\{i\}}^{n_i} \, \mathcal{H}^{j,\leq\mu}_{G,I,W} \big)\Big|_{\overline{\eta_I}}$$

where v_i are closed points of X (chosen such that $\times_{i\in I} v_i$ is included in the smooth locus of $\mathcal{H}^{j,\leq\mu}_{G,I,W}$) and $\mathscr{H}_{G,v_i}$ is the local Hecke algebra on v_i.

By the Eichler-Shimura relations, the sum is in fact over a finite number of $(n_i)_{i\in I}$. Thus each $\mathfrak{M}_{\mu}$ is a module of finite type over a Hecke algebra.

By Drinfeld's lemma (Hecke-version), we prove Proposition 1.

Idea of the proof of Proposition 2, we need a lemma :
for any geometric point $\overline{x}$ of $(\overline{\eta})^I$ and any specialisation map $\overline{\eta_I} \to \overline{x}$ in $(\overline{\eta})^I$, the induced morphism

$$\mathcal{H}^j_{G,I,W}\Big|_{\overline{x}} \to \mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$$

is an isomorphism, i.e. $\mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^I}$ is ind-smooth.

The proof of this lemma is very similar to V. Lafforgue's proof of the fact that $\mathcal{H}^j_{G,I,W}\Big|_{\Delta(\overline{\eta})} \to \mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$ is an isomorphism (which uses the Eichler-Shimura relations).

Then, Proposition 1 implies that $\mathcal{H}^j_{G,I,W}\Big|_{(\overline{\eta})^I}$ is constant.

Smoothness

Theorem 3

The $\mathbb{Q}_\ell$-sheaf $\mathcal{H}^j_{G,I,W}$ is ind-smooth over X^I.

Ind-smooth means an inductive limit of smooth (i.e. lisse) $\mathbb{Q}_\ell$-sheaves. Equivalently, for any geometric points $\overline{x}$, $\overline{y}$ of X^I and any specialisation map $\overline{x} \to \overline{y}$, the induced morphism $\mathcal{H}^j_{G,I,W}\Big|_{\overline{y}} \to \mathcal{H}^j_{G,I,W}\Big|_{\overline{x}}$ is an isomorphism.

Remark : if $\mathrm{Cht}_{G,I,W}$ is proper (for example : [Eike Lau, On degenerations of $\mathscr{D}$-shtukas]), then $\mathcal{H}^j_{G,I,W}$ is a constructible $\mathbb{Q}_\ell$-sheaf. We know that $\mathcal{H}^j_{G,I,W}$ is a smooth $\mathbb{Q}_\ell$-sheaf over X^I.

Corollary

The action of $\mathrm{Weil}(\eta, \overline{\eta})^I$ on $\mathcal{H}^j_{G,I,W}\Big|_{\overline{\eta_I}}$ factors through $\mathrm{Weil}(X, \overline{\eta})^I$

Proof of smoothness : example of I singleton

Let $I = \{1\}$ be a singleton. Let W be a representation of $\widehat{G}$. We have a cohomology sheaf $\mathcal{H}^j_{G,\{1\},W}$ over X.

For any geometric point $\overline{v}$ of X (over a closed point v) and any specialization map $\mathfrak{sp} : \overline{\eta} \to \overline{v}$, we have an induced morphism

$$\mathfrak{sp}^* : \mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{v}} \to \mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{\eta}}$$

We want to prove that $\mathfrak{sp}^*$ is an isomorphism. This is equivalent to say that $\mathcal{H}^j_{G,\{1\},W}$ is ind-smooth over X.

Idea : construct an inverse of $\mathfrak{sp}^*$ using some creation and annihilation operators and Proposition 2.

Construction of a morphism $\mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{\eta}} \to \mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{v}}$

Let α be the composition of the morphisms :

$$\begin{array}{c}
\mathcal{H}^j_{G,\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\big|_{\overline{v}} \\
\Big\downarrow \mathcal{C}^{\sharp,\{2,3\}}_{\delta} \quad \text{creation operator} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})} \\
\Big\downarrow \mathfrak{sp}^*_{\{2\}} \quad \text{canonical morphism} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}} \\
\Big\downarrow \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}} \quad \text{annihilation operator} \\
\mathbb{Q}_\ell\big|_{\overline{\eta}} \otimes \mathcal{H}^j_{G,\{3\},W}\Big|_{\overline{v}}
\end{array}$$

Construction of the morphism $\mathfrak{sp}^*_{\{2\}}$

If $\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ with $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ $\mathbb{Q}_\ell$-sheaves over X, then $\mathfrak{sp}^*_{\{2\}}$ is just

$$\mathcal{F}_1\big|_{\overline{\eta}} \otimes \mathcal{F}_2\big|_{\overline{v}} \otimes \mathcal{F}_3\big|_{\overline{v}} \xrightarrow{\mathrm{Id}\otimes\mathfrak{sp}^*\otimes\mathrm{Id}} \mathcal{F}_1\big|_{\overline{\eta}} \otimes \mathcal{F}_2\big|_{\overline{\eta}} \otimes \mathcal{F}_3\big|_{\overline{v}}$$

In general, similar to Proposition 2, using the Eichler-Shimura relations and Drinfeld's lemma (Hecke-version) we show that the restriction of $\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}$ to the schemes $\overline{\eta}\times\overline{\eta}\times\overline{\eta}$, $\quad \overline{\eta}\times\overline{\eta}\times\overline{v}$, $\quad \overline{\eta}\times\overline{v}\times\overline{\eta}$ and $\overline{v}\times\overline{\eta}\times\overline{\eta}$ are constant sheaves. Then using a technical lemma, we construct the morphism $\mathfrak{sp}^*_{\{2\}}$.

Reminder about the "Zorro" lemma

Note that the composition

$$W \otimes \mathbb{Q}_\ell \xrightarrow{Id \otimes \delta} W \otimes W^* \otimes W \xrightarrow{\mathrm{ev} \otimes Id} \mathbb{Q}_\ell \otimes W$$

is the identity.

By the functoriality, we have

"Zorro" lemma

The composition of morphisms of sheaves over X :

$$\mathcal{H}^j_{\{1\},W} \otimes \mathbb{Q}_\ell \xrightarrow{\mathcal{C}_\delta^{\sharp,\{2,3\}}} \mathcal{H}^j_{\{1,2,3\},W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(X)} \xrightarrow{\mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}}} \mathbb{Q}_\ell \otimes \mathcal{H}^j_{\{3\},W}$$

is the identity.

Proof of $\alpha \circ \mathfrak{sp}^* = \mathrm{Id}$

The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{H}^j_{\{1\},W}\Big|_{\overline{v}} \otimes \mathbb{Q}_\ell\big|_{\overline{v}} & \xrightarrow{\quad \mathfrak{sp}^* \quad} & \mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\big|_{\overline{v}} \\
\Big\downarrow \mathcal{C}_\delta^{\sharp,\{2,3\}} & & \Big\downarrow \mathcal{C}_\delta^{\sharp,\{2,3\}} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{v})} & \xrightarrow{\mathfrak{sp}^*_{\{1\}}} & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})} \\
 & \searrow^{\mathfrak{sp}^*_{\{1,2\}}} & \Big\downarrow \mathfrak{sp}^*_{\{2\}} \\
\Big\downarrow \mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}} & & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}} \\
 & & \Big\downarrow \mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}} \\
\mathbb{Q}_\ell\big|_{\overline{v}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}} & \xrightarrow[\simeq]{\quad \mathrm{Id} \quad} & \mathbb{Q}_\ell\big|_{\overline{\eta}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}}
\end{array}
$$

The composition of the right vertical morphisms is α. By "Zorro" lemma, the composition of the left vertical morphisms is the identity.

Proof of $\mathfrak{sp}^* \circ \alpha = \mathrm{Id}$

The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} & \xrightarrow[\simeq]{\qquad\qquad} & \mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{\eta}} \\
\Big\downarrow {\scriptstyle \mathcal{C}^{\sharp,\{2,3\}}_{\delta}} & & \Big\downarrow {\scriptstyle \mathcal{C}^{\sharp,\{2,3\}}_{\delta}} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times_{\overline{\mathbb{F}_q}}\Delta^{\{2,3\}}(\overline{v})} & \xrightarrow{\mathfrak{sp}^*_{\{2,3\}}} & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{\eta})} \\
\Big\downarrow {\scriptstyle \mathfrak{sp}^*_{\{2\}}} & \nearrow {\scriptstyle \mathfrak{sp}^*_{\{3\}}} & \Big\downarrow {\scriptstyle \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times_{\overline{\mathbb{F}_q}}\overline{v}} & & \\
\Big\downarrow {\scriptstyle \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}}} & & \\
\mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}} & \xrightarrow{\mathfrak{sp}^*} & \mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{\eta}}
\end{array}
$$

The composition of the left vertical morphisms is α. By "Zorro" lemma, the composition of the right vertical morphisms is the identity.

Some general remarks

1. When there is a level structure $N \subset X$, the cohomology sheaf $\mathcal{H}^{j}_{G,N,I,W}$ is ind-smooth over $(X \smallsetminus N)^I$.

2. The same argument of smoothness works for any reductive group over F. (The constant term morphisms are only for split groups for the moment.)

3. The same argument works for cohomology with $\mathbb{Z}_\ell$-coefficients (in the place of $\mathbb{Q}_\ell$-coefficients).

4. Remark of Gaitsgory and Varshavsky : using the smoothness of $\mathcal{H}^{j}_{G,I,W}$ and the constant term morphisms, we can prove that when μ is big enough, $\mathcal{H}^{j,\leq\mu}_{G,I,W}$ is smooth over X^I.

5. We have

$$\mathrm{Rep}(\widehat{G}^I) \to \text{Ind-Const}(X^I), \quad W \mapsto \mathcal{H}_{G,I,W}$$

By the smoothness property, we have $\mathrm{Rep}(\widehat{G}^I) \to \text{Ind-Lisse}(X^I)$. This is used in the proof of

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \xrightarrow{\sim} C_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$

and

$$\mathrm{Tr}(\mathrm{Frob}_* \circ \mathrm{Hecke}_{I,W}, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \xrightarrow{\sim} H_{G,I,W}$$

in [Arinkin-Gaitsgory-Kazhdan-Raskin-Rozenblyum-Varshavsky].

6. When there is a level structure N, we have (example with I singleton)

$$\begin{array}{ccccccc}
\mathrm{Cht}_{N,\{1\},W} & \xrightarrow{\pi} & \mathrm{Cht}_{\{1\},W}\Big|_{X \smallsetminus N} & \longrightarrow & \mathrm{Cht}_{\{1\},W} & \longleftarrow & \mathrm{Cht}_{\{1\},W}\Big|_{v} \\
& \searrow & \downarrow{\scriptstyle \mathfrak{p}} & & \downarrow{\scriptstyle \mathfrak{p}} & & \downarrow{\scriptstyle \mathfrak{p}} \\
& & X \smallsetminus N & \longrightarrow & X & \longleftarrow & v \in N
\end{array}$$

We hope to prove that for $\pi_! \,\mathrm{Sat}_{N,\{1\},W}$, the nearby cycles commute with $\mathfrak{p}_!$ (in progress).