Hodge, p-adic², and tropical iterated integrals

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Everything joint with

Sasha Shmakov

(archimedean setting)

and

Eric Katz

(p-adic/tropical setting)

Problem:

Complex iterated integrals are not single-valued

Ex $a, b \in \mathbb{C}^*$

- $$\int_a^b \frac{dz}{z} = \log(b) - \log(a) + 2\pi i \cdot n$$

- $a, b \in \mathbb{C} \setminus \{0, 1\}$

$$\sum \frac{z^n}{n^2}$$

$$\int_a^b \left(\int_a^{z_1} \frac{dz_2}{1 - z_2} \right) \frac{dz_1}{z_1} = \mathrm{Li}_2(b) - \mathrm{Li}_2(a) + \cdots$$

linear comb. of terms involving $\log(a), \log(b)$, $\log(1-a)$, $\log(1-b)$, $2\pi i$, $(2\pi i)^2$

Formal Defn

$\gamma : [0,1] \to X$ ← manifold

$\omega_1, \dots, \omega_n$: C^∞-1-forms on X

$$\int_\gamma \omega_1 \cdots \omega_n = \int_{0 \le t_1 \le t_2 \le \cdots \le t_n \le 1} \gamma^* \omega_1 \wedge \cdots \wedge \gamma^* \omega_n$$

Problem: This value depends on γ.

X - smooth cpx variety

$A^1(X)$ = C^∞ - 1-forms on X

<u>Defn</u> $\Omega \in \bigoplus_{i \geq 0} A^1(X)^{\otimes i}$ is htpy-invt

if $\int_\gamma \Omega$ only depends on

based htpy class of γ, for all γ

<u>Ex</u> • X - Riemann surface

all tensors of holomorphic forms are htpy invt

• X arbitrary

$(\omega_{12}, \omega_1 \otimes \omega_2)$ s.t. $d\omega_{12} = \omega_1 \wedge \omega_2$

Thm (L. - Shmakov)

X smooth complex variety

Ω -htpy inv. $\exists$ single-valued iterated integral $\int_a^b$: htpy-inv forms $\to \mathbb{C}$ (compare to Deligne, Francis Brown)

$$\int_a^b \Omega \in \mathbb{C}$$ (if wt$(H^1(X)) = 2$)

such that in general non-zero

(1) $\int_a^b \Omega$ varies real-analytically in a, b

(2) (functoriality) $f: X \to Y$ $\quad \int_{f(a)}^{f(b)} \Omega = \int_a^b f^* \Omega$

(3) (composition)

$$\int_a^b \omega_1 \otimes \cdots \otimes \omega_n = \sum_i \int_a^c \omega_1 \otimes \cdots \otimes \omega_i \int_c^b \omega_{i+1} \otimes \cdots \otimes \omega_n$$

... (many other properties)

Examples

(1) $\int_a^b \frac{dz}{z} = 2\log\left|\frac{b}{a}\right|$

$$\int_a^b \left(\frac{dz}{z}\right)^{\otimes N} = \frac{2^N}{N!}\log^N\left|\frac{b}{a}\right|$$

(2) $\int_a^b \left(\frac{dz}{1-z}\right)\otimes\left(\frac{dz}{z}\right)$ = long expression in terms of Bloch-Wigner dilogarithm

$$D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$$

(3) Hain's elliptic dilogarithm

$E\setminus pt$

Cor X sm. complex variety, iterated extn $(\mathcal{O}, d)$

$(\mathcal{E}, \nabla: \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$ unipotent flat v.b.

Then there is (canonical, functorial) real-analytic trivialization

$$\mathcal{E}^{\text{real-analytic}} \xrightarrow{\sim} \underline{\mathcal{E}_x}^{\text{real-analytic}}$$

Idea of constr.: MHS on $\pi_1(X)^{unip}$:

- $\mathbb{C}[\pi_1(X,x)]$, $\mathbb{C}[\pi_1(X;x,y)]$

  group ring of $\pi_1(X,x)$ — free v.s. on $\pi_1(X;x,y)$

- $J \subseteq \mathbb{C}[\pi_1(X,x)]$ - augmentation ideal

- $\mathbb{C}[[\pi_1(X,x)]]$, $\mathbb{C}[[\pi_1(X;x,y)]]$

  J-adic completion of $\mathbb{C}[\pi_1]$ — J-adic completion of $\mathbb{C}[\pi_1(X;x,y)]$

- $F^\bullet$, $W_\bullet$ - Hodge + Weight filtration

  Weight filtration: J-adic filtration for X proper,

  $F^\bullet \mathbb{C}[[\pi_1(X,x)]]^\vee$ - count dz's

  Chen: space of htpy inv't iterated integrals

(Deligne)

Prop $\exists!\ p(x,y) \in \mathbb{C}[[\pi_1(X;x,y)]]$

s.t.

already appear in work of Brown, enough if $X = \mathbb{P}^1 \setminus D$

(1) $p(x,y) = 1 \mod \mathcal{J}$

(2) $p(x,y) \in F^0\mathbb{C}[[\pi_1(X;x,y)]]$

(3) $\overline{p(x,y)} \in \bigcap_{N \geq 1} (F^{N-1} + W_{-N-1})$

Prop
- $p(x,x) = 1$
- $f_*(p(x,y)) = p(f(x), f(y))$
- $p(x,y) \circ p(y,z) = p(x,z)$.

$\vdots$
- $p(x,y)$ is group-like.

$\Delta(p(x,y)) = p(x,y) \otimes p(x,y)$

Defn

$$\int_a^b \Omega := \int_{\overline{p(a,b)}} \Omega$$

why this conjugate

$p(a,b) \in F^0 \Rightarrow \int_{p(a,b)} \omega = 0$

if ω is holomorphic.

p-adic setting

X - curve / K - p-adic field

- X - has good reduction:

  Coleman iterated integrals are single-valued b/c $\exists$ canonical Frobenius-invt "path"

- X - bad reduction

  Berkovich iterated integrals not single-valued

  b/c depends on a choice of path on the dual graph of reduction

Vologodsky: $\exists$ canonical

path b/w any 2 pts

even in bad reduction

<u>Q</u> How to compute/integrate?

<u>A</u> tropical iterated integral

Structure

$X/\mathcal{O}_K$ – proper semi-stable w/ smooth generic fiber

$$\pi(X; x, y)^{\text{log-crys}} := \mathcal{O}^{\vee}_{\pi_1(X_k; \bar{x}, \bar{y})^{\text{log-crys, unip}}}$$

- $\varphi, N \in \mathrm{End}(\pi(X; x, y)^{\text{log-crys}})$
- $W_\bullet$ – weight filtration
- $gr_i^W \pi(X; x, y)$ – satisfy wt-monodromy conj. (Betts-L.)

$$\pi(X; x, y)^{dR} := \mathcal{O}^{\vee}_{\pi_1(X_K; x, y)^{dR, unip}}$$

- $F^\bullet$ – Hodge filtration
- $W^\bullet$ – weight filtration

Comp: $\pi^{\text{log-crys}} \otimes_{\mathrm{Frac}\, W(k)} K_{st} \xrightarrow{\sim} \pi^{dR} \otimes_K K_{st}$

Coleman integration:

<u>Prop</u> X - good reduction

$$\exists!\, p(x,y) \in \Pi(X; x,y)^{crys} \quad \text{s.t.}$$

- $p(x,y) \equiv 1 \mod W_{-1}$
- $\varphi(p(x,y)) = p(x,y)$

<u>Prop</u> Canonical iso

$$(J^{n+1})^\vee \cap F^n \mathcal{O}_{\pi_1(X_K; x,y)^{dR,un,p}}$$

$$\downarrow \wr$$

$$\Omega \in H^1(X_K, \Omega^1_{X_K})^{\otimes n}$$

<u>Defn</u> $\displaystyle\int_{x,\,\text{Coleman}}^{y} \Omega := \langle p(x,y), \Omega \rangle$

Berkovich integration

$X/\mathcal{O}_K$ - semistable curve

Γ - dual graph of special fiber

Prop

$$(\pi^{\text{log-crys}})^{\rho} \subseteq \pi(X;x,y)^{\text{log-crys}} \xrightarrow{} \mathrm{Frac}\,W(k)[[\pi_1(\Gamma;x,y)]]$$

has canonical section given by

$$(\pi^{\text{log-crys}})^{\rho}$$

Defn Given $p \in K_{st}[[\pi_1(\Gamma;x,y)]]$

$$\int_{p,\,\text{Berkovich}} \Omega := \langle \rho, \Omega \rangle$$

Vologodsky integration

Prop (Vologodsky)

$\exists!\ p_{can} \in \Pi(X; x, y)^{log\text{-}crys}$ s.t.

(1) $p_{can} = 1 \mod W_{-1}$

(2) $\varphi(p_{can}) = p_{can}$

(3) For all $n > 0$, $N^n(p_{can}) \in W_{-n-1}$

Pf Follows from wt-monodromy for $\Pi^{log\text{-}crys}$.

Defn

$$\int_{x, \text{Vologodsky}}^{y} \Omega := \langle p_{can}, \Omega \rangle$$

Q How to compute ρ_{can}?

A tropical iterated integrals
→ Cheng-Katz (combinatorial versions)

Γ - graph

$\mathcal{H}(\Gamma) :=$ harmonic 1-forms on Γ

$:= f : E^{\text{oriented}}(\Gamma) \longrightarrow K$ s.t.

$\forall v \in \Gamma$

$$\sum_{\substack{e \text{ adjacent to} \\ v, \text{ oriented} \\ \text{outward}}} f(e) = 0.$$

Defn (Cheng-Katz, tropical iterated integrals)

- $\displaystyle\int_{e,\mathrm{trop}} \omega_1 \otimes \cdots \otimes \omega_n = \frac{1}{n!}\prod_i \omega_i(e)$ for $e \in E(\Gamma)$
- $\displaystyle\int_{p_1 \circ p_2,\mathrm{trop}} \omega_1 \otimes \cdots \otimes \omega_n = \sum_{i=0}^{n} \left(\int_{p_1} \omega_1 \otimes \cdots \otimes \omega_i\right)\left(\int_{p_2} \omega_{i+1} \otimes \cdots \otimes \omega_n\right)$

Thm (Katz-L.)

Under the canonical identification

$$\left(\Pi(X; x, y)^{\text{log-crys}}\right)^{\varphi} = \mathrm{Frac}(W(k))[[\pi_1(\Gamma; \bar{x}, \bar{y})]]$$

p_{vol} is sent to the unique element such that

$$\int_{p_{vol},\mathrm{trop}} \omega_1 \otimes \cdots \otimes \omega_n = 0 \quad \text{for all } n > 0,$$

ω_i.

$p_{vol} \equiv 1 \mod J$

Pf idea

Re-interpret tropical iterated integration in terms of monodromy action on $\pi^{\text{log-crys}}$

Prop $M_\bullet$ - monodromy filtration on $\pi(X;x,y)^{\text{log-crys}}$

$\exists$ canonical identification

$$M_{-2i}\left(\pi(X;x,y)^{\text{log-crys}}/W_{-i}\right) \simeq \left(\mathcal{H}(\Gamma)^{\otimes i}\right)^{\vee}$$

Thm Given

$$p \in \operatorname{Frac}(W(k))[\pi_1(\Gamma;\bar{x},\bar{y})] = \left(\pi(X;x,y)^{\text{log-crys}}\right)^{\varphi}$$

and $\omega_1 \otimes \cdots \otimes \omega_n \in \mathcal{H}(\Gamma)^{\otimes n}$, we have

$$\int_{p,\text{trop}} \omega_1 \otimes \cdots \otimes \omega_n = \langle N^n(p), \omega_1 \otimes \cdots \otimes \omega_n \rangle$$

Cor Algorithm for computing Vologodsky integrals in terms of Berkovich integrals.