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(joint work w/ Tong Liu).

§1. Background

k = perfect field of char. $p>0$. $K/W(k)[\frac{1}{p}]$ ~~totally~~ ramified of degree e.

We're interested in $\rho: \mathrm{Gal}_K \longrightarrow GL_n(\mathbb{Q}_p)$ cont.

Instead $\tilde{\rho}: \mathrm{Gal}_K \longrightarrow GL_n(\mathbb{Z}_p) \twoheadrightarrow GL_n(\mathbb{Z}/p^n)$.

Fact: Fix $r \in \mathbb{N}$, s.t. $e \cdot r < p-1$.

Then $\{\tilde{\rho}$ s.t. $\tilde{\rho}[\frac{1}{p}]$ is crystalline (semistable) w/ Hodge–Tate weights in $[0,r]\}$

1:1 $\updownarrow$

f.g. S-module

$\{$Breuil modules of height $r\}$.

$(M, \mathrm{Fil}^r M, \varphi_r: \mathrm{Fil}^r M \longrightarrow M, \nabla: M \longrightarrow M)$ + cond.

"divided Frob." "connection"

sub-S-mod of M

More notations: $\mathfrak{S} := W(k)[[u]] \xrightarrow{\theta} \mathcal{O}_K$, $u \longmapsto \pi$

$\ker(\theta) = (E(u))$ $E \equiv u^e \mod p$.

$\varphi: \mathfrak{S} \longrightarrow \mathfrak{S}$, $W(k) \xrightarrow{\varphi} W(k)$, $u \longmapsto u^p$

S = p-cpltd PD envelope of θ $= \left(\mathfrak{S}\left[\frac{(u^e)^n}{n!}\right]\right)^\wedge$.

Rmk: Also version w/ torsion coeff. e.g. $\mathrm{Gal}_K \circlearrowright$ f.g. $\mathbb{Z}_p$-mod. Δ.

§2. Geometric origin.

$\mathcal{X} \xrightarrow{\text{sm. proper}} \mathrm{Spec}(\mathcal{O}_K) \hookrightarrow \mathrm{Spec}(S)$, then $H^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p)$ is such $\tilde{\rho}$ $(i \le r)$.

Thm (Caruso): ① If $e\cdot(i+1) < p-1$, then.

$$H^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p) \longleftrightarrow \left(H^i_{crys}(\mathcal{X}/S), H^i_{crys}(\mathcal{X}/S, \mathcal{J}^{[i]}), \varphi_i, \nabla\right).$$

(similar (derived) mod p^n version holds as well).

② If $e\cdot i < p-1$, then.

$$H^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{F}_p) \longleftrightarrow \left(H^i_{crys}(\mathcal{X}/S, \mathcal{O}/p), H^i_{crys}(\mathcal{X}/S, \mathcal{J}^{[i]}/p), \varphi_i, \nabla\right)$$

Rmk: ① $H^i(\mathcal{J}^{[i]}) \longrightarrow H^i_{crys}(\mathcal{X}/S)$ (analogue of H–dR degeneration)

② $H^i_{crys}(\mathcal{X}/S, \mathcal{O})$ "looks like" a $\mathbb{Z}_p$-module.

③ admissibility.

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Thm. (L.–Liu) If $e \cdot i < p-1$, then

$$H^i_{ét}(\mathcal{X}_{\bar{K}}, \mathbb{Z}/p^n) \longleftrightarrow \left(H^i_{crys}(\mathcal{X}/S, \mathcal{O}/p^n), \text{—"—}\right)$$

⌈ Caruso's strategy: Step 1. Understand mod p situation extremely well.

Step 2. Induction. $R\Gamma/p \longrightarrow R\Gamma/p^m \longrightarrow R\Gamma/p^{m-1} \xrightarrow{+1}$.

say $m \geq 2$;

$$\underbrace{H^{i-2}_{crys}(\mathcal{O}/p)}_{\checkmark} \xrightarrow{\delta} \underbrace{H^{i-1}_{crys}(\mathcal{O}/p)}_{\checkmark} \longrightarrow H^{i-1}_{crys}(\mathcal{O}/p^2) \longrightarrow \underbrace{H^{i-1}_{crys}(\mathcal{O}/p)}_{\checkmark} \xrightarrow{0} \underline{\underline{H^i_{crys}(\mathcal{O}/p)}}$$ ⌋

Upshot: H^{i+1} is "still kinda nice".

of what??

§3. Prismatic cohomology theory.

Thm (Bhatt–Morrow–Scholze; Bhatt–Scholze).

Associated w/ $\begin{pmatrix} \mathcal{X} \\ \downarrow \\ \mathcal{O}_K \xleftarrow{\theta} \mathfrak{S} \end{pmatrix}$ is $(R\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}), \varphi)$

perfect $\mathfrak{S}$-complex.

(1) $R\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \overset{L}{\underset{\mathfrak{S},\varphi}{\otimes}} S \cong R\Gamma_{crys}(\mathcal{X}/S)$ compatible with φ.

$\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \hookrightarrow S$

(2)

$$\begin{tikzcd} H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \arrow[r, "\exists\varphi"] \arrow[rr, bend right, "\cdot E^i"'] & H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \arrow[r, "\varphi"] \arrow[rr, bend left, "\cdot E^i"] & H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \arrow[r, "\exists\psi"] & H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \end{tikzcd}$$

(generalized Kisin module of height i).

$\mathfrak{M}^i := H^i_{\Delta}(\mathcal{X}/\mathfrak{S})$. $\quad 0 \longrightarrow \mathfrak{M}^i \underset{\mathfrak{S},\varphi}{\otimes} S \longrightarrow H^i_{crys}(\mathcal{X}/S) \longrightarrow \mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}^{i+1}, {}_{\varphi}S) \longrightarrow 0.$

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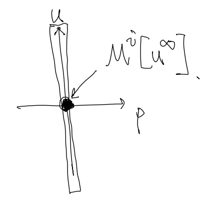
Lemma: $\mathcal{M}^i_{tor} = \mathcal{M}^i[p^\infty]$.

pf: From Iwasawa theory: $P :=$ char. poly. of $\mathcal{M}^i_{tor}$.

$\varphi(p)$ and P differ by power of E.

$\overset{\text{Exercise.}}{\Longrightarrow} P \sim p^n$.

Lemma: $\mathcal{M}^i_{tor} / \mathcal{M}^i[u^\infty]$ is successive extn of $\mathfrak{S}/p = k[[u]]$.



Prop. (Min): if $e \cdot i < p-1$, then $\mathcal{M}^i[u^\infty] = 0$.

Toy case: $\mathcal{M}^i[u^\infty] = k[u]/u^\alpha$.

$$\begin{tikzcd} k[u]/u^{p\cdot\alpha} \arrow[r, "\varphi"] \arrow[rr, bend right, "u^{e\cdot i}"'] & k[u]/u^{\alpha} \arrow[r, "\psi"] & k[u]/u^{p\cdot\alpha} \end{tikzcd}$$

$\Longrightarrow \quad u^{p\cdot\alpha} \mid (u^{e\cdot i} \cdot u^\alpha) \iff \alpha \cdot p \leq \alpha + ei \iff ei \geq \alpha \cdot (p-1)$.

$\Rightarrow \alpha = 0$.

§4. Put together.

Thm (L.-Liu) $i < p-1$, TFAE:

① $(H^i_{crys}(-), \ldots)$ is a Breuil mod. of height i

② H^i_{Δ} & H^{i+1}_{Δ} are u-torsion free.

Moreover when this happens, the Breuil mod $\longleftrightarrow H^i_{ét}$.

Rmk. mod p^n version holds as well.

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pf. ② ⟹ ①: $H^i_{\Delta} \otimes_{\mathfrak{S},\varphi} S \cong H^i_{crys}(\mathfrak{X}/S)$ ✓

Filtrations: Thm (L.–Liu):

$$\left(R\Gamma_{\Delta} \otimes_{\mathfrak{S},\varphi} \mathfrak{S}, \mathrm{Fil}^{\bullet}_N\right) \overset{L}{\underset{(\mathfrak{S}, E^{\bullet})}{\otimes}} \left(S, E^{[\bullet]}\right) \cong \left(R\Gamma_{crys}(\mathfrak{X}/S), R\Gamma_{crys}(\mathcal{J}^{[\bullet]})\right).$$

Cor. $l \le p$, $R\Gamma_{\Delta} \otimes_{\mathfrak{S},\varphi} \mathfrak{S} / \mathrm{Fil}^l_N \cong R\Gamma_{crys}(\mathfrak{X}/S, \mathcal{O}/\mathcal{J}^l)$.

take $l=i$

u-torsion free

$$\begin{array}{ccccccc} H^{i-1}(\quad) & \xrightarrow{0} & H^i(\mathrm{Fil}^i_N) & \hookrightarrow & H^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} \mathfrak{S} \\ \downarrow \cong & & \downarrow & & \downarrow \\ H^{i-1}(\quad) & \xrightarrow{0} & H^i(\mathcal{J}^{[i]}) & \hookrightarrow & H^i_{crys} \end{array}$$

① ⟹ ② + comparing ét w/ Breuil mod:

throw all known specialization at it

+ Thm of Antieau–Mathew–Morrow–Nikolaus

on p-adic nearby cycles.

Prop. $T = \max\{r \mid e \cdot r < p-1\}$, then up to degree $T+1$.

$H^*_{\Delta}(\mathfrak{X}/\mathfrak{S}, \mathcal{O}_{\Delta}/p^m)$ are u-torsion free.

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Toy case: $* = T+1$, $m=2$.

$$H^T_\Delta(\mathcal{O}/p) \xrightarrow{0} H^{T+1}_\Delta(\mathcal{O}/p) \longrightarrow H^{T+1}_\Delta(\mathcal{O}/p^2) \longrightarrow H^{T+1}_\Delta(\mathcal{O}/p)$$

$\mathrm{coker}\left(H^T(\mathcal{O}/p) \longrightarrow H^{T+1}(\mathcal{O}/p)\right)$ has no u-torsion.

toy case:

$$\begin{tikzcd} k[[u]] \arrow[r, "\varphi"] \arrow[d, "u^{p\cdot\alpha}"] & k[[u]] \arrow[d, "u^{\alpha}"] \\ k[[u]] \arrow[r, "\varphi"] & k[[u]] \end{tikzcd}$$

at least $u^{\alpha\cdot(p-1)}$.

but it also factors $u^{e\cdot T}$

$e\cdot T < p-1 \Rightarrow \alpha = 0$.